THEOREM: THE CONSTANT RULE

The derivative of a constant function is zero. That is, if c is a real number,

then $$\frac{d}{dx}[c]=0$$

Example 1: Find the derivative of the function $g(x)=-5$.

$g'(x)=0$

THEOREM: THE POWER RULE

If n is a rational number, then the function $f(x)=x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$$

For f to be differentiable at $x=0$, n must be a number such that x^{n-1} is defined on an interval containing zero.

Example 2: Find the following derivatives.

a. $f(x)=x^{-5}$

$f'(x)=-5x^{-6}$

$\boxed{f'(x)=-\frac{5}{x^6}}$

b. $f(x)=x^{1/2}$

$f'(x)=1/2x^{-1/2}$

$=\boxed{\frac{1}{2\sqrt{x}}}$

c. $f(x)=x^{-2/3}$

$f'(x)=-\frac{2}{3}x^{-5/3}$

$=\boxed{-\frac{2}{3\sqrt[3]{x^5}}}$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}\left[cf(x)\right]=cf'(x)$$

Example 3: Find the slope of the graph of $f(x)=2x^3$ at

$f'(x)=2\cdot 3x^2$

$f'(x)=6x^2$

a. $x=2$

$f'(2)=6(2)^2$

$\boxed{f'(2)=24}$

b. $x=-6$

$\boxed{f'(-6)=216}$

c. $x=0$

$\boxed{f'(0)=0}$

THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$$

Example 4: Find the equation of the line tangent to the graph of $f(x)=x-\sqrt{x}$ at x = 4.

$f'(x)=1-\frac{1}{2}x^{-1/2}$

$f'(4)=1-\frac{1}{2\sqrt{4}}$

$f'(4)=1-\frac{1}{4}$

$\boxed{f'(4)=\frac{3}{4}}$

THEOREM: DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\sin x]=\cos x \qquad \frac{d}{dx}[\cos x]=-\sin x$$

$$\frac{d}{dx}[\csc x]=-\csc x\cot x \qquad \frac{d}{dx}[\sec x]=\sec x\tan x$$

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\cot x]=-\csc^2 x$$

Example 5: Find the derivative of the following functions:

a. $f(x)=\dfrac{\sin x}{6}$

$f'(x)=\frac{1}{6}(\cos x)$

$=\dfrac{\cos x}{6}$

b. $r(\theta)=5x-3\cos\theta$

$r'(\theta)=5+3\sin\theta$

c. $f(x)=x\tan x$

$f'(x)=1\tan x+x\sec^2 x$

$\boxed{f'(x)=\tan x+x\sec^2 x}$

d. $r(\theta)=\dfrac{\cos\theta}{\cot\theta}$

TRIG Identities!

OR $r(\theta)=\dfrac{\cos\theta}{\left(\frac{\cos\theta}{\sin\theta}\right)}=\sin\theta$

$r'(\theta)=\dfrac{\cot\theta(-\sin\theta)-[\cos\theta(-\csc^2\theta)]}{\cot^2\theta}$

$r'(\theta)=\dfrac{-\cot\theta\sin\theta+\cos\theta\csc^2\theta}{\cot^2\theta}$

$r'(\theta)=\cos\theta$

THEOREM: THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the derivative of the first function times the second function, plus the first function times the derivative of the second function.

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

This rule extends to cover products of more than two factors. For example the derivative of the product of functions $fghk$ is

$$\frac{d}{dx}[fghk]=f'(x)g(x)h(x)k(x)+f(x)g'(x)h(x)k(x)+f(x)g(x)h'(x)k(x)+f(x)g(x)h(x)k'(x)$$

Example 6: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $g(x)=x\cos x$

$g'(x)=(1)(\cos x)+(x)(-\sin x)$

$=\boxed{\cos x-x\sin x}$

b. $h(t)=\left(3-\sqrt{t}\right)^2$

u^2

$u = 3-\sqrt{t}$

$du/dy = -\frac{1}{2\sqrt{t}}$

$2u \; du/dy$

$2u \cdot -\frac{1}{2\sqrt{t}}$

$-\frac{u}{\sqrt{t}} = \boxed{-\frac{(3-\sqrt{t})}{\sqrt{t}}}$

c. $f(x)=\left(x^3-x\right)\left(x^2+2\right)\left(x^2+x-1\right)$

$$\boxed{f'(x)=(3x^2-1)(x^2+2)(x^2+x-1)+(x^3-x)(2x)(x^2+x-1)+(x^3-x)(x^2+2)(2x+1)}$$

THEOREM: THE QUOTIENT RULE

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x)\neq 0$. Moreover, the derivative of f/g is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 7: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $g(x)=x^4\left(1-\frac{2}{x+1}\right) = x^4\left(1-2(x+1)^{-1}\right)$

$g'(x)=4x^3\left(1-2(x+1)^{-1}\right)+x^4\left(0+2(x+1)^{-2}(1)\right)$

$g'(x) = 4x^3-8x^3(x+1)^{-1}+2x^4(x+1)^{-2}$

$g'(x) = 2x^3(x+1)^{-2}\left[2(x+1)^2-4(x+1)+x\right]$

$$g'(x) = \frac{2x^3}{(x+1)^2}\left[2(x^2+2x+1) - 4x - 4 + x\right]$$

$$g'(x) = \frac{2x^3}{(x+1)^2}\left[2x^2+4x+2-3x-4\right]$$

$$\boxed{g'(x) = \frac{2x^3(2x^2+x-2)}{(x+1)^2}}$$

b. $h(s)=\dfrac{s}{\sqrt{s}-1}$

$$h'(s)=\frac{1(\sqrt{s}-1)\frac{2\sqrt{s}}{2\sqrt{s}}-s\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}-1)^2}$$

$$h'(s)=\frac{2s-2\sqrt{s}-s}{2\sqrt{s}(\sqrt{s}-1)^2}$$

$$h'(s)=\frac{\cancel{s}^{\sqrt{s}}-2\sqrt{\cancel{s}}^{1}}{2\cancel{\sqrt{s}}(\sqrt{s}-1)^2}$$

$$\boxed{h'(s)=\frac{\sqrt{s}-2}{2(\sqrt{s}-1)^2}}$$

c. $f(x)=\tan x$

$$\boxed{f'(x)=\sec^2 x}$$

Example 8: Find the derivative of the trigonometric functions.

a. $g(x)=-2\csc x$

$$g(x)=(-2)\left(\frac{1}{\sin x}\right) \quad g'(x)=(-2)\left[\frac{-\cos x}{\sin^{2} x}\right]=\boxed{+2\cot x\csc x}$$

$f'=0 \quad g'=\cos x$

OR $g'(x)=-2(-\csc x\cot x)$

$=\boxed{2\csc x\cot x}$

b. $h(t)=\cot^2 t$

$\csc^4 t$

$h(t)=(\cot t)^2$

$h'(t)=2(\cot t)'(-\csc^2 t) \rightarrow \boxed{h'(t)=-2\csc^2 t\cot t}$

c. $r(s)=\dfrac{\sec s}{s}$

$$r'(s)=\frac{(\sec s\tan s)(s)-(\sec s)(1)}{s^2}$$

$$\boxed{r'(s)=\frac{\sec s(s\tan s-1)}{s^2}}$$

Alternate way

$r(s)=s^{-1}\sec s$

$r'(s)=-s^{-2}\sec s+s^{-1}\sec s\tan s$

$$\boxed{r'(s)=\frac{-\sec s+s\tan s}{s^2}}$$

Example 9: Find the given higher-order derivative.

a. $f''(x)=2-\frac{2}{x}, \quad f'''(x)$

$f''(x)=2-2x^{-1}$

$f'''(x)=2x^{-2}$

$\boxed{f'''(x)=\frac{2}{x^2}}$

b. $f^{(4)}(x)=2x+1, \quad f^{(6)}(x)$

$\boxed{f^{(6)}(x)=0}$

Theorem: The Chain Rule

If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x, then $y=f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \text{ or } \frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x).$$

Example 10: Find the derivative using the Chain Rule.

a. $y=(2-x)^3$

$\frac{dy}{dx}=3(2-x)^2(-1)$

$\boxed{\frac{dy}{dx}=-3(2-x)^2}$

b. $f(x)=\sin 2x$

$f'(x)=(\cos 2x)\cdot 2$

$\boxed{f'(x)=2\cos 2x}$

c. $h(t)=\frac{\sqrt{t}}{\sqrt{t}-1}=t^{1/2}(t^{1/2}-1)^{-1}$

$h'(t)=\frac{1}{2}t^{-1/2}(t^{1/2}-1)^{-1}+t^{1/2}\left[-(t^{1/2}-1)^{-2}\cdot\frac{1}{2}t^{-1/2}\right]$

$h'(t)=\frac{1}{2}t^{-1/2}(t^{1/2}-1)^{-2}\left[(t^{1/2}-1)-t^{1/2}\right]$

$h'(t)=\boxed{-\frac{1}{2\sqrt{t}(\sqrt{t}-1)^2}}$

Theorem: The General Power Rule

If $y=[u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx}=n[u(x)]^{n-1}\cdot\frac{du}{dx} \text{ or } \frac{d}{dx}[u^n]=nu^{n-1}u'.$$

Example 11: Find the derivative of the following functions.

a. $y=\sec x$

$\boxed{\frac{dy}{dx}=\sec x\tan x}$

b. $y=\sec 2x$

$\frac{dy}{dx}=\sec(2x)\tan(2x)\cdot 2$

$\boxed{\frac{dy}{dx}=2\sec 2x\tan 2x}$

c. $y = \sec^2 x = (\sec x)^2$

$\frac{\partial y}{\partial x} = 2(\sec x)^1 \sec x \tan x$

$\boxed{\frac{\partial y}{\partial x} = 2\sec^2 x \tan x}$

d. $y = \sec x^2$

$\frac{\partial y}{\partial x} = \sec(x^2)\tan(x^2) \cdot 2x$

$\boxed{\frac{\partial y}{\partial x} = 2x \sec x^2 \tan x^2}$

e. $y = x^5$

f. $y = (2x^3 - 5)^5$

g. $y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2} x^{-1/2}$

$\boxed{y' = \frac{1}{2\sqrt{x}}}$

h. $y = \sqrt{\cos x} = \cos^{1/2} x$

$y' = \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$

$\boxed{y' = -\frac{\sin x}{2\sqrt{\cos x}}}$

i. $f(x) = x^2(2-x)^{2/3}$

j. $f(x) = \sqrt{\dfrac{1}{2x^3+15}}$

k. $h(x) = x\sin^2 4x$

l. $f(x) = \cot\sqrt[3]{x} - \sqrt[3]{\cot x}$

Example 12: Find the equation of the tangent line at $t=1$ for the function

$$s(t)=\left(9-t^2\right)^{2/3}.$$

Theorem: Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x]=\dfrac{1}{x}$

2. $\dfrac{d}{dx}[\ln u]=\dfrac{u'}{u},\quad u>0$

Example 13: Find the derivative.

$(\ln x)^3 = \ln x \cdot \ln x \cdot \ln x$

$\ln x^3 = \ln(x \cdot x \cdot x)$

$= \ln x + \ln x + \ln x$

$= 3\ln x$

a. $y=(\ln x)^3$

$y' = 3(\ln x)^2 \cdot \frac{1}{x}$

$\boxed{y' = \dfrac{3(\ln x)^2}{x}}$

b. $f(x)=\ln|\cos x|$

$f'(x) = \dfrac{-\sin x}{\cos x}$

$\boxed{f'(x) = -\tan x}$

c. $h(x)=\ln x^x = x\ln x$

$h'(x) = 1\ln x + x\left(\frac{1}{x}\right)$

$\boxed{h'(x) = 1+\ln x}$

Theorem: Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}\left[e^x\right]=e^x$

2. $\dfrac{d}{dx}\left[e^u\right]=e^u u'$

Example 14: Find the derivative.

a. $y = xe^{-x}$

$y' = (1)(e^{-x}) + (x)(-1)(e^{-x})$

$\boxed{y' = e^{-x}(1-x)}$

b. $f(x) = e^{\sin 2x}$

$\boxed{f'(x) = 2\cos 2x \, e^{\sin 2x}}$

c. $h(t) = \dfrac{e^t}{\ln e^{\sqrt{t}}} = \dfrac{e^t}{t^{1/2}\ln e} = \dfrac{e^t}{t^{1/2}}$

$$h'(t) = \frac{e^t t^{1/2}\frac{2\sqrt{t}}{2\sqrt{t}} - e^t \cdot \frac{1}{2\sqrt{t}}}{\left(t^{1/2}\right)^2}$$

$$h'(t) = \frac{2te^t - e^t}{(2\sqrt{t})(t)}$$

$$\boxed{h'(t) = \frac{e^t(2t-1)}{2t^{3/2}}}$$

Theorem: Derivatives for Bases other than *e*

Let a be a positive real number $(a \neq 1)$ and let u be a differentiable function of x.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u u'$
3. $\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{(\ln a)u}$

Example 15: Find the derivative.

a. $y = 2^{3x}$

$y' = \left(2^{3x}\right)(3)(\ln 2)$

$\boxed{y' = (\ln 8)2^{3x}}$

b. $f(x) = \log 5x$

$f'(x) = \left(\frac{5}{5x}\right) \cdot \frac{1}{\ln 10}$

$\boxed{f'(x) = \frac{1}{x \ln 10}}$

c. $h(t) = \frac{\log_3 t^2}{\sin t}$