MATH 251/GRACEY NAME________________________

PRACTICE/CHAPTER 8

1. Find the integral.

a. $\int \frac{(\ln x)^2}{x}\,dx = \boxed{\frac{(\ln x)^3}{3} + C}$

$u = \ln x$

$du = \frac{dx}{x}$

$$\begin{array}{r} 1 + \frac{1}{x^2-1} \\ (x^2-1)\overline{\big)\, x^2 + 0x + 0} \\ -(x^2 - 1) \\ \hline 1 \end{array}$$

b. $\int \ln\sqrt{x^2-1}\,dx = \frac{1}{2}\int \ln(x^2-1)\,dx$

IBP

$u = \ln(x^2-1)$

$du = \frac{2x}{x^2-1}\,dx$

$dv = dx$

$v = x$

$= \frac{1}{2}\left[x\ln(x^2-1) - \int x\left(\frac{2x}{x^2-1}\right)dx\right]$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2\int \frac{x^2\,dx}{x^2-1}\right]$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2\left\{\int 1\,dx + \int \frac{dx}{x^2-1}\right\}\right]$

$x^2 - 1 = \left(\sqrt{x^2-1}\right)^2$

trig sub

$x = 1\sec\theta$, $dx = \sec\theta\tan\theta\,d\theta$

$x^2 - 1 = \sec^2\theta - 1$

$x^2 - 1 = \tan^2\theta$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2x - 2\int \frac{\sec\theta\,\cancel{\tan\theta}\,d\theta}{\tan^{\cancel{2}}\theta}\right]$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2x - 2\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta}\,d\theta\right]$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2x - 2\int \csc\theta\,d\theta\right]$

$= \frac{1}{2}\left[x\ln(x^2-1) - 2x + 2\ln|\csc\theta + \cot\theta|\right] + C$

$$= \boxed{\frac{1}{2}\left[x\ln(x^2-1) - 2x + 2\ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right|\right] + C}$$

c. $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

$x = 2\tan\theta$, $x^3 = 8\tan^3\theta$

$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta}$

$= \sqrt{4(1+\tan^2\theta)}$

$= 2\sec\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{(8\tan^3\theta)(\cancel{2}\sec^{\cancel{2}}\theta\,d\theta)}{\cancel{2\sec\theta}}$$

$$= 8\int \tan^2\theta \cdot \sec\theta\tan\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$

$$= 8\left[\int (\sec\theta)^2 \cdot \sec\theta\tan\theta\,d\theta - \int \sec\theta\tan\theta\,d\theta\right]$$

$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$

$$= 8\left[\left(\frac{\sqrt{4+x^2}}{2}\right)^3 \cdot \frac{1}{3} - \frac{\sqrt{4+x^2}}{2}\right] + C$$

$$= \boxed{\frac{(4+x^2)^{3/2}}{3} - 4\sqrt{4+x^2} + C}$$

d. $\displaystyle\int \frac{1}{1-\cos\theta}\,d\theta$

$$\frac{1}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \frac{1+\cos\theta}{1-\cos^2\theta}$$

$$= \frac{1+\cos\theta}{\sin^2\theta}$$

$$= \csc^2\theta + \csc\theta\cot\theta$$

$$= \int (\csc^2\theta + \csc\theta\cot\theta)\,d\theta$$

$$= \boxed{-\cot\theta - \csc\theta + C}$$

e. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx$

f. $\int \frac{x-28}{x^2-x-6}\,dx$

2. Evaluate the definite integral.

a. $\int_0^{\pi/4} \tan^3 x dx$

b. $\int_0^2 e^{-x} \cos x dx$

c. $\int_0^e \ln x^2 dx$

d. $\int_1^\infty \frac{1}{x^2+5} dx$

3. For the following limits:

    a. Describe the type of indeterminate form (if any) that is obtained by direct substitution.

    b. Evaluate the limit, using L'Hôpital's Rule if necessary.

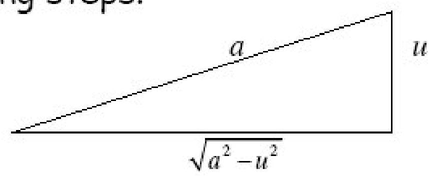
        i. $\lim_{x\to\infty} x \ln x$

        ii. $\lim_{x\to 0} \dfrac{\arcsin x}{x}$

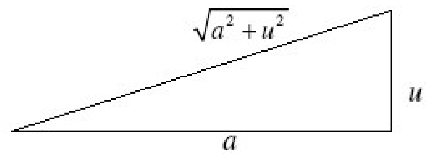
TRIGONOMETRIC SUBSTITUTION ($a > 0$)

Let $f(c) = 0$ where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

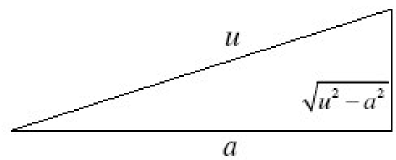
1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$.



Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$, where $0 \le \theta < \dfrac{\pi}{2}$ or $\dfrac{\pi}{2} < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.

Indeterminate Forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty - \infty,\ 0 \cdot \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$
2. If f is continuous on the interval $(-\infty,b]$, then
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$
3. If f is continuous on the interval $(-\infty,\infty)$, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx,\ c \text{ is any real number.}$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at *b*, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$
2. If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at *a*, then
$$\int_c^b f(x)\,dx = \lim_{c\to a^+}\int_a^c f(x)\,dx$$
3. If f is continuous on the interval $[a,b]$, except for some *c* in (a,b) at which *f* has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.