3/2/11

- warm up
- lecture 8.3

Friday

8.4

→ Trig Substition

Warm-up

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}\,dx = \frac{-x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int \frac{2xe^{x^2}\cancel{(x^2+1)}\,dx}{\cancel{(x^2+1)}} = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$u = x^2 e^{x^2}$

$\frac{du}{dx} = 2xe^{x^2} + x^2 \cdot 2xe^{x^2}$

$du = 2xe^{x^2}(x^2+1)\,dx$

$\int dv = \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx$

$v = \frac{1}{2(x^2+1)}$

$$= \frac{-x^2 e^{x^2} + e^{x^2}(x^2+1)}{2(x^2+1)} + C$$

$$= \frac{\cancel{-x^2 e^{x^2}} + \cancel{x^2 e^{x^2}} + e^{x^2}}{2(x^2+1)} + C$$

$$= \boxed{\frac{e^{x^2}}{2(x^2+1)} + C}$$

possible dvs:

$xe^{x^2}\,dx$ or

$\frac{x}{(x^2+1)^2}\,dx$

$$\int \ln x\,dx = x\ln x - \int \cancel{x}\left(\frac{dx}{\cancel{x}}\right)$$

$$= x\ln x - x + C$$

$$= \boxed{-x(1-\ln x) + C}$$

$u = \ln x$

$du = \frac{dx}{x}$

$dv = dx$

$v = x$

$u = \tan x \rightarrow du = \sec^2 x\,dx$

$u = \sec x \rightarrow du = \sec x \tan x\,dx$

$u = \sin x \rightarrow du = \cos x\,dx$

$u = \cos x \rightarrow du = -\sin x\,dx$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = 1 - \cos^2 x$

$\tan^2 x = \sec^2 x - 1$

$\sec^2 x = \tan^2 x + 1$

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$$

$$\int \cos^2 x \sin x\,dx = -\int \cos^2 x\,(-\sin x\,dx) = -\frac{\cos^3 x}{3} + C$$

$u = \cos x$

$du = -\sin x\,dx$

$$\int \cos^3 x\,dx = \int \cos^2 x \cdot \cos x\,dx$$

$$= \int (1-\sin^2 x)\cos x\,dx$$

$$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx$$

$$= \boxed{\sin x - \frac{\sin^3 x}{3} + C}$$

$$\int \cos^4 x\,dx = \int \left[\cos^2 x\right]^2 dx$$

$$= \int \left[\frac{1+\cos 2x}{2}\right]^2 dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\left[x + \sin 2x + \int \frac{1+\cos 4x}{2}\,dx\right]$$

$$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + C$$

$$= \boxed{\frac{1}{32}\left(12x + 8\sin 2x + \sin 4x\right) + C}$$

MATH 251/GRACEY NAME ______________________________

WORKSHEET/8.3

TRIGONOMETRIC INTEGRALS

We will study techniques for evaluating integral of the form

$$\int \sin^m x \cos^n x\,dx \quad \text{and} \quad \int \sec^m x \tan^n x\,dx$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING SINE AND COSINE

1. If the power of the **sine** is **odd and positive**, save one sine factor and convert the remaining factors to cosines. Then expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{odd}}} x \cos^n x\,dx = \int \overbrace{\left(\sin^2 x\right)^k}^{\substack{\text{convert to cosines}\\ \text{using pythagorean}\\ \text{identity}}} \cos^n x \overbrace{\sin^1 x\,dx}^{\text{save for } du}$$

$$= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x\,dx$$

2. If the power of the **cosine** is **odd and positive**, save one cosine factor and convert the remaining factors to sines. Then expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{odd}}} x\,dx = \int \sin^m x \overbrace{\left(\cos^2 x\right)^k}^{\substack{\text{convert to sines}\\ \text{using pythagorean}\\ \text{identity}}} \overbrace{\cos^1 x\,dx}^{\text{save for } du}$$

$$= \int \sin^m x \left(1-\sin^2 x\right)^k \cos x\,dx$$

3. If the powers of **both** the sine and cosine are **even and nonnegative**, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

To convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING SECANT AND TANGENT

1. If the power of the **secant** is **even and positive**, save one secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{even}}} x \tan^n x\,dx = \int \overbrace{\left(\sec^2 x\right)^{k-1}}^{\text{convert to tangents using pythagorean identity}} \tan^n x \underbrace{\sec^2 x\,dx}_{}^{} $$

$$\text{save for } du: \ \sec^2 x\,dx$$

$$= \int \left(1+\tan^2 x\right)^{k-1} \tan^n x \sec^2 x\,dx$$

2. If the power of the **tangent** is **odd and positive**, save one secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{odd}}} x\,dx = \int \overbrace{\sec^{m-1} x\left(\tan^2 x\right)^k}^{\text{convert to secants using pythagorean identity}} \overbrace{\sec^1 x \tan x\,dx}^{\text{save for } du}$$

$$= \int \sec^m x\left(\sec^2 x - 1\right)^k \sec x \tan x\,dx$$

3. If there are **no secant factors** and the power of the **tangent** is **even and positive**, convert a tangent-squared factor to a secant-squared factor, then expand and repeat, if necessary.

$$\int \tan^n x\,dx = \int \tan^{n-2} x \overbrace{\left(\tan^2 x\right)}^{\text{convert to secants using pythagorean identity}} dx$$

$$= \int \tan^{n-2} x\left(\sec^2 x - 1\right) dx$$

4. If the integral is of the form $\int \sec^m x dx$, where *m* is **odd and positive**, use **integration by parts**!
5. If none of these techniques work, try converting to sines and cosines.

INTEGRALS INVOLVING SINE-COSINE PRODUCTS WITH DIFFERENT ANGLES

$$\sin(mx)\sin(nx) = \frac{1}{2}\left(\cos\left[(m-n)x\right] - \cos\left[(m+n)x\right]\right)$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\left(\sin\left[(m-n)x\right] + \sin\left[(m+n)x\right]\right)$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left(\cos\left[(m-n)x\right] + \cos\left[(m+n)x\right]\right)$$

$$\cos(mx)\sin(nx) = \frac{1}{2}\left(\sin\left[(m-n)x\right] - \sin\left[(m+n)x\right]\right)$$

1. Integrate.

a. $\int \cos^7 x dx = \int (\cos^2 x)^3 \cos^1 x \, dx$

$= \int (1-\sin^2 x)^3 \cos x \, dx$

$= \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\cos x \, dx$

$= \int \cos x \, dx - 3\int \sin^2 x \cos x \, dx + 3\int \sin^4 x \cos x \, dx$

$- \int \sin^6 x \cos x \, dx$

$= \boxed{\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{\sin^7 x}{7} + C}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$(1-\sin^2 x)^3 = \left[1 + (-\sin^2 x)\right]^3$$

$a=1,\ b=-\sin^2 x$

$$\left[1+(-\sin^2 x)\right]^3 = 1(1)^3(-\sin^2 x)^0 + 3(1)^2(-\sin^2 x)^1 + 3(1)^1(-\sin^2 x)^2 + 1(1)^0(-\sin^2 x)^3$$

$$= 1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x$$

$$\sin^4 3x = (\sin^2 3x)^2$$

$$= \left(\frac{1-\cos 6x}{2}\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos 6x + \cos^2 6x\right)$$

$$= \frac{1}{4} - \frac{1}{2}\cos 6x + \frac{1}{4}\,\frac{(1+\cos 12x)}{2}$$

(see above)

b. $\int \sin^4 3x\,dx = \int\left(\frac{1}{4} - \frac{1}{2}\cos 6x + \frac{1}{8} + \frac{1}{8}\cos 12x\right)dx$

$= \int\left(\frac{3}{8} - \frac{1}{2}\cos 6x + \frac{1}{8}\cos 12x\right)dx$

$= \frac{3}{8}x - \frac{1}{2}\frac{\sin 6x}{6} + \frac{1}{8}\cdot\frac{\sin 12x}{12} + C$

$= \boxed{\frac{3}{8}x - \frac{1}{12}\sin 6x + \frac{1}{96}\sin 12x + C}$

c. $\int \sec^5 x\,dx$

d. $\int \tan^5 2x \sec 2x dx$ $= \int (\tan^2 2x)^2 \sec 2x \tan 2x \, dx$

$= \int (\sec^2 2x - 1)^2 \sec 2x \tan 2x \, dx$

$= \int (\sec^4 2x - 2\sec^2 2x + 1) \overbrace{\sec 2x \tan 2x \, dx}$

$= \int \left(\frac{2\sec^4 2x}{2} - 2\sec^2 2x\right) \sec 2x \tan 2x \, dx$

$+ \frac{1}{2}\int 2\sec 2x \tan 2x \, dx$

$= \frac{\sec^5 2x}{10} - \frac{\sec^3 2x}{3} + \frac{\sec 2x}{2} + C$

e. $\int \tan^4 x \sec^2 x dx$ $= \int u^4 \cancel{\sec^2 x} \left(\frac{du}{\cancel{\sec^2 x}}\right)$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \frac{u^5}{5} + C$

$= \boxed{\frac{\tan^5 x}{5} + C}$

f. $\int \sin(3x)\cos(2x)\,dx$

g. $\int \tan^{6}\left(\frac{x}{2}\right)dx$