3/30/11

- warm up
using warm-up
from the 10.3 W.S.
- Lecture 10.3

Friday

- Lecture 10.4
- Last day to drop w/a W

* Tomorrow is a holiday - no classes!

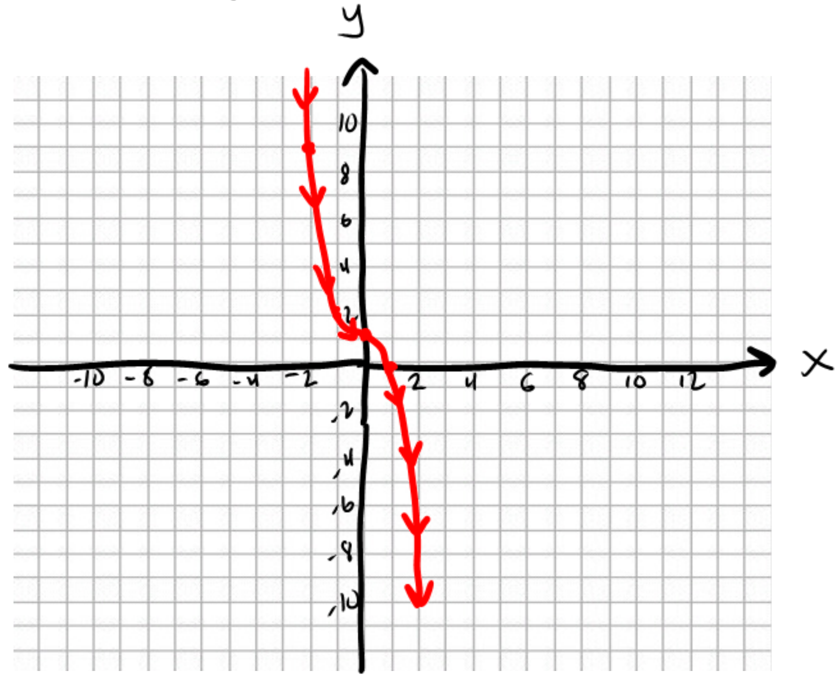
FRIDAY 4/8 is EXAM 4
10.1-10.5

Warm-up:

1. Consider the parametric equation $x=\sqrt[3]{t}$ and $y=1-t$.

   a. Graph the parametric equation, indicating the orientation.

t	x	y	(x,y)
-8	-2	9	(-2, 9)
-1	-1	2	(-1, 2)
0	0	1	(0, 1)
1	1	0	(1, 0)
8	2	-7	(2, -7)



   b. Eliminate the parameter.

$(x)^3=(\sqrt[3]{t})^3$

$x^3=t$

$y=1-t$

$t=1-y$

$x^3=1-y$

$y=1-x^3$

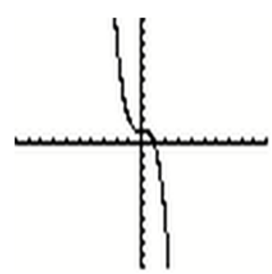
Domain: $(-\infty,\infty)$

Range: $(-\infty,\infty)$

• The range of $x(t)$ is the domain of the plane curve → $(-\infty,\infty)$

• The range of $y(t)$ is the range of the plane curve → $(-\infty,\infty)$

mode



   c. Evaluate $\frac{dy}{dx}$ using your result from part b.

$$\frac{dy}{dx}=\frac{d}{dx}(1-x^3)$$

$$\frac{dy}{dx}=-3x^2$$

d. Now evaluate the following derivatives using the original parametric equations, $x=\sqrt[3]{t}$ and $y=1-t$.

i. $\frac{dx}{dt} = \frac{1}{3}t^{-2/3} = \boxed{\frac{1}{3t^{2/3}}}$

ii. $\frac{dy}{dt} = \boxed{-1}$

e. Now evaluate $\frac{dy}{dx}$ using your results from part d.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{\left(\frac{1}{3t^{2/3}}\right)} = \boxed{-3t^{2/3}}$$

But is this the same?!

$\frac{dy}{dx} = -3t^{2/3}$ and $x^3 = t$

so $\frac{dy}{dx} = -3\left(x^3\right)^{2/3}$

$= -3x^{3(2/3)}$

$= -3x^2$

which is the same!

PARAMETRIC FORM OF THE DERIVATIVE

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope at C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

2. Find $\frac{dy}{dx}$ for the curve given by $x = \cos t$ and $y = \sin t$.

$\frac{dy}{dt} = \cos t$

$\frac{dx}{dt} = -\sin t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t}$$

$$\frac{dy}{dx} = \boxed{-\cot t}$$

a. Now evaluate the second derivative, that is $\frac{d}{dx}\left[\frac{dy}{dx}\right]$. $= \frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-\cot t)}{-\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t}$$

$$\frac{d^2y}{dx^2} = \csc^2 t\,(-\csc t)$$

$$\frac{d^2y}{dx^2} = \boxed{-\csc^3 t}$$

HIGHER ORDER DERIVATIVES OF PARAMETRIC EQUATIONS

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope at C at (x, y) is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

In general we have,

$$\frac{d^ny}{dx^n} = \frac{\frac{d}{dt}\left[\frac{d^{n-1}y}{dx^{n-1}}\right]}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

3. Determine the t intervals on which the curve is concave downward or concave upward for the curve given by $x = t + 1$ and $y = t^2 + 3t$.

$\frac{dy}{dt} = 2t + 3$, $\frac{dx}{dt} = 1$

1st deriv.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dy}{dx} = \frac{2t+3}{1}$

$\frac{dy}{dx} = 2t+3$

2nd deriv

$\frac{d^2y}{dx^2} = \frac{\partial/\partial t\left(\partial y/\partial x\right)}{\partial x/\partial t}$

$\frac{d^2y}{dx^2} = \frac{\partial/\partial t\left(2t+3\right)}{1}$

$\frac{d^2y}{dx^2} = 2$

Normally we'd set the 2nd deriv. equal to zero, solve for critical numbers and then test the intervals defined by the crit. #'s to see if they + or − using the 2nd deriv. funct.

Here $\frac{d^2y}{dx^2} = 2 > 0$ all the time.

So the curve is concave upwards for all $t \rightarrow (-\infty, \infty)$

ARC LENGTH IN PARAMETRIC FORM

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\,dt.$$

NOTE: When applying the arc length formula to a curve, be sure that the curve is traced only once on the interval of integration.

4. Find the arc length of the curve given by

$$x = \arcsin t \text{ and } y = \ln\sqrt{1-t^2} \text{ on the interval } 0 \le t \le \frac{1}{2}.$$

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{1}{\sqrt{1-t^2}}\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{1-t^2}$$

$$y = \frac{1}{2}\ln(1-t^2)$$

$$\left(\frac{dy}{dt}\right)^2 = \left(\frac{1}{\cancel{2}}\left(\frac{-\cancel{2}t}{1-t^2}\right)\right)^2$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{t^2}{(1-t^2)^2}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1(1-t^2)}{(1-t^2)(1-t^2)} + \frac{t^2}{(1-t^2)^2}$$

$$= \frac{1-t^2+t^2}{(1-t^2)^2}$$

$$= \frac{1}{(1-t^2)^2}$$

So... $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{1}{(1-t^2)^2}}$

$$= \frac{1}{1-t^2}$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$s = \int_0^{1/2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

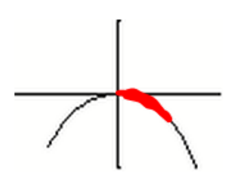
$$s = \int_0^{1/2} \frac{1}{1-t^2}\,dt$$

$$s = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| \Bigg|_0^{1/2}$$

$$s = \frac{1}{2}\left[\ln\left|\frac{3/2}{1/2}\right| - \ln\left|\frac{1}{1}\right|\right]$$

$$s = \frac{1}{2}(\ln 3 - 0)$$

$$\boxed{s = \ln\sqrt{3} \text{ units}}$$



AREA OF A SURFACE OF REVOLUTION

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, does not cross itself on the interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ Revolution about the $x-$axis: $g(t) \ge 0$

2. $S = 2\pi\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ Revolution about the $y-$axis: $f(t) \ge 0$

5. Find the area of the surface generated by revolving the curve below about the y- axis.

$$x = \frac{1}{3}t^3 \text{ and } y = t + 1 \text{ on the interval } 1 \le t \le 2$$

$f(t) = \frac{1}{3}t^3$

$\frac{dx}{dt} = t^2$, $\frac{dy}{dt} = 1$

$$S = 2\pi\int_a^b f(t)\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

$$S = 2\pi\int_1^2 \frac{1}{3}t^3\sqrt{(t^2)^2 + (1)^2}\,dt$$

$u = t^4 + 1$

$\frac{du}{dt} = 4t^3$

$dt = \frac{du}{4t^3}$

$$S = \frac{2\pi}{4 \cdot 3}\int_1^2 4t^3\sqrt{t^4 + 1}\,dt$$

$$S = \frac{\pi}{\cancel{6}\,3}\left(\frac{\cancel{2}}{3}(t^4+1)^{3/2}\right|_1^2$$

$$\boxed{S = \frac{\pi}{9}\left(17^{3/2} - 2^{3/2}\right) \text{ units}^2 \quad \text{or} \quad S = \frac{\pi}{9}\left(17\sqrt{17} - 2\sqrt{2}\right) \text{ units}^2}$$