MATH 251/GRACEY NAME________________________

WORKSHEET/8.8

IMPROPER INTEGRALS

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2. If f is continuous on the interval $(-\infty,b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3. If f is continuous on the interval $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,\ c \text{ is any real number.}$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.

1. Determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

a. $\int_0^{\infty}(x-1)e^{-x}dx$ $= \lim_{b\to\infty}\int_0^b (x-1)e^{-x}\,dx$

$\lim_{b\to\infty}\frac{-b}{e^b} = \lim_{b\to\infty}\frac{-1}{e^b} = 0$

So $\int_0^{\infty}(x-1)e^{-x}\,dx$ converges to 0

Consider the indefinite integral first

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$\int (x-1)e^{-x}\,dx = -e^{-x}(x-1) + \int e^{-x}\,dx = -e^{-x}(x-1) - e^{-x} = -e^{-x}(x-1+1)$

$= -xe^{-x}$

$u = x-1 \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

So $-xe^{-x}\Big|_0^b = -be^{-b} - 0$

$= -be^{-b} = \frac{-b}{e^b}$

b. $\int_{1}^{\infty} \frac{5}{x}\,dx$

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at *b*, then $\int_{a}^{b} f(x)\,dx = \lim_{c\to b^-} \int_{a}^{c} f(x)\,dx$
2. If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at *a*, then $\int_{c}^{b} f(x)\,dx = \lim_{c\to a^+} \int_{a}^{c} f(x)\,dx$
3. If f is continuous on the interval $[a,b]$, except for some *c* in (a,b) at which *f* has an infinite discontinuity, then
$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.

2. Determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

a. $\displaystyle\int_{-1}^{2} \frac{dx}{x^3}$

b. $\displaystyle\int_{0}^{6} \frac{4}{\sqrt{6-x}}\,dx$

Theorem: A Special Type of Improper Integral

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

3. Evaluate the definite integral.

a. $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$

b. $\int_1^{\infty} \frac{5}{\sqrt[5]{x^6}} dx$

c. $\int_0^{2} (2-t)\sqrt{t}\,dt$