4/11/11

- lecture 9.1 sequences

Wednesday

- 9.2

Friday

9.3

Exams not ready
→ maybe not 'til after spring break

Sequences

General form is a_n = some sequence defined with the variable in subscript as the input

Domain of a sequence usually starts with 1, and is an integer

Domain: $\{1, 2, 3, \ldots\}$ or $\{n \mid n \in \mathbb{Z}^+\}$

Range can be all real numbers. It depends on how the sequence is defined.

$\{a_n\}$ represents all possible outputs → known as terms

$\{a_n\} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$

Factorials → product of decreasing factors

$n! = n(n-1)(n-2)\cdots(2)(1)$

$5! = 5\cdot4\cdot3\cdot2\cdot1$

$$\frac{n!}{(n-1)!} = \frac{n\cdot\cancel{(n-1)!}}{\cancel{(n-1)!}} = \boxed{n}$$

consider $\dfrac{5!}{(5-1)!} = \dfrac{5!}{4!} = \dfrac{5\cdot\cancel{4!}}{\cancel{4!}} = \boxed{5}$

$$\frac{(2n)!}{[2(n+1)]!} = \frac{(2n)!}{(2n+2)!}$$

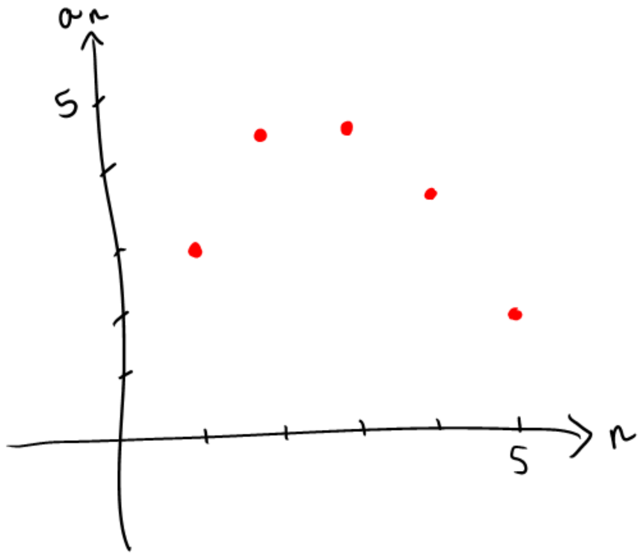
$$= \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

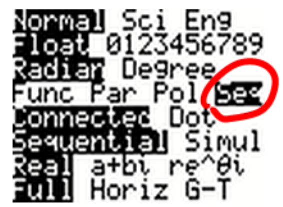
$$= \boxed{\frac{1}{(2n+2)(2n+1)}}$$

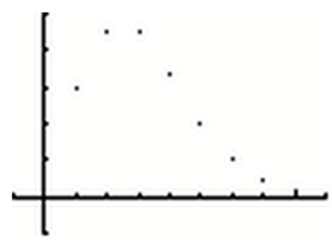
$n = 5$

$$\frac{10!}{[2(6)]!} = \frac{10!}{12!} = \frac{\cancel{10!}}{12 \cdot 11 \cdot \cancel{10!}} = \frac{1}{12 \cdot 11}$$

Graph the first 5 points of the sequence $a_n = \dfrac{3^n}{n!}$







Definition

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

or if its terms are nonincreasing

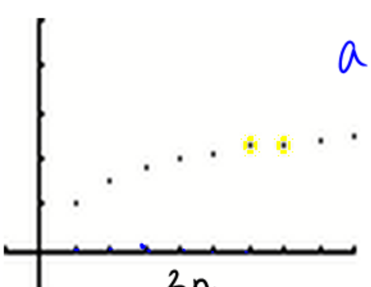
$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$$

Bounded below at 1

Bounded above at

$\lim_{n \to \infty} \frac{3n}{n+2} = 3$, so a_n is bounded

$a_n = \frac{3n}{n+2}$ has nondecreasing terms, so it is monotonic



$a_n = \dfrac{3n}{n+2}$

check a_6 and a_7

$a_6 = \dfrac{3(6)}{(6)+2} = \dfrac{18}{8} = \dfrac{9}{4}$ is $\dfrac{9}{4} \le \dfrac{7}{3}$ yes

$a_7 = \dfrac{3(7)}{(7)+2} = \dfrac{21}{9} = \dfrac{7}{3}$ $\quad \dfrac{27}{12} \le \dfrac{28}{12}$

Since a_n is bounded and monotonic, it converges

Definition of a bounded Sequence

① A sequence $\{a_n\}$ is bounded above if there's a real number M such that $a_n \le M$ for all n. The number M is called the upper bound of the sequence.

② A sequence $\{a_n\}$ is bounded below if there's a real number N such that $N \le a_n$ for all n. The number N is called the lower bound of the sequence.

③ A sequence is bounded if it is bounded above and bounded below.

Limit of a Sequence

Let L be a real number. The <u>limit</u> of a sequence $\{a_n\}$ is L, written $\lim_{n\to\infty} a_n = L$ if for each $\varepsilon > 0$ there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. *Very important

If the limit L of a sequence does exist, then the sequence converges to L. If the limit of a sequence does not exist, then the sequence diverges.

Theorem: Limit of a sequence

Let L be a real number. Let f be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$.

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$

Theorem: Absolute value theorem

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \qquad \text{then} \qquad \lim_{n\to\infty} a_n = 0$$

1. Write the first 5 terms of the sequence.

a. $a_n = \dfrac{3^n}{n!}$

$a_1 = \dfrac{3^1}{(1)!} = 3$

$a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2}$

$a_3 = \dfrac{3^3}{3!} = \dfrac{27}{6} = \dfrac{9}{2}$

$a_4 = \dfrac{3^4}{4!} = \dfrac{81}{24} = \dfrac{27}{8}$

$a_5 = \dfrac{3^5}{5!} = \dfrac{\cancel{3}\cdot 3^4}{5\cdot 4\cdot \cancel{3}\cdot 2\cdot 1} = \dfrac{81}{40}$

$$\boxed{\{a_5\} = \left\{3, \frac{9}{2}, \frac{9}{2}, \frac{27}{8}, \frac{81}{40}\right\}}$$

b. $a_n = \dfrac{2n}{n+3}$

$a_1 = \dfrac{2}{4} = \dfrac{1}{2}$

$a_2 = \dfrac{4}{5}$

$a_3 = \dfrac{6}{6} = 1$

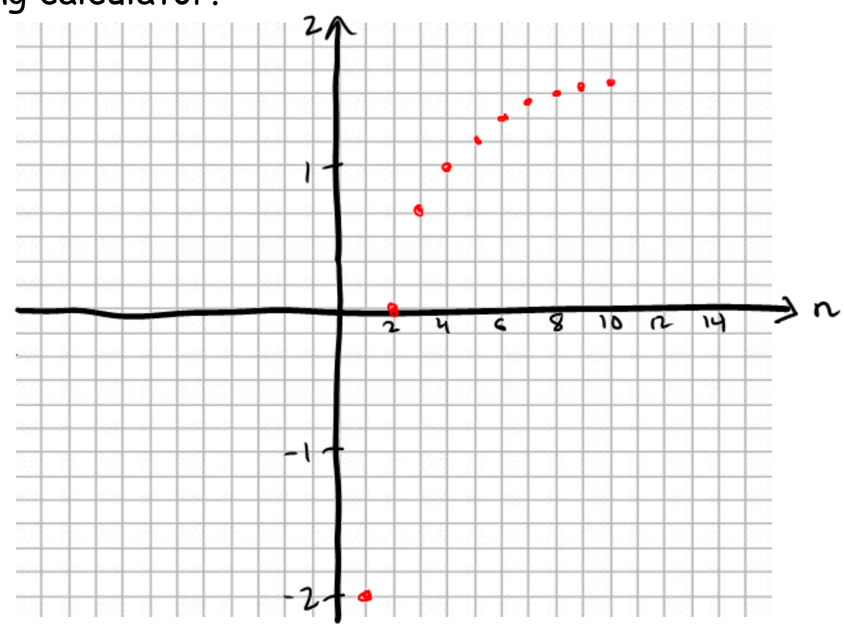
$a_4 = \dfrac{8}{7}$

$a_5 = \dfrac{10}{8} = \dfrac{5}{4}$

$$\boxed{\{a_5\} = \left\{\frac{1}{2}, \frac{4}{5}, 1, \frac{8}{7}, \frac{5}{4}\right\}}$$

2. Graph the first 10 terms of the sequence $a_n = 2 - \dfrac{4}{n}$ by hand and then check your result using a graphing calculator.

n	a_n	(n, a_n)
1	-2	$(1, -2)$
2	0	$(2, 0)$
3	$2/3 \approx .67$	$(3, 2/3)$
4	1	$(4, 1)$
5	$6/5 \approx 1.2$	$(5, 6/5)$
6	$4/3 \approx 1.33$	$(6, 4/3)$
7	$10/7 \doteq 1.4$	$(7, 10/7)$
8	$3/2 = 1.5$	$(8, 3/2)$
9	$14/9 \approx 1.55$	$(9, 14/9)$
10	$8/5 \approx 1.6$	$(10, 8/5)$



3. Simplify the ratio of factorials.

a. $\frac{25!}{22!} = \frac{25 \cdot 24 \cdot 23 \cdot \cancel{22!}}{\cancel{22!}}$

$= \boxed{13{,}800}$

b. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot \cancel{n!}}{\cancel{n!}}$

$= \boxed{n+1}$

4. Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find its limit.

a. $a_n = 1 + (-1)^n$

$a_1 = 0$, $a_2 = 2$, $a_3 = 0$ oscillates

$\lim_{n\to\infty} \left[1 + (-1)^n\right]$ does not exist, so, by definition, a_n diverges.

b. $a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

$f(n) = a_n$, for all n

$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1} = \lim_{n\to\infty} \frac{n^{1/3}/n^{1/3}}{\frac{n^{1/3}}{n^{1/3}} + \frac{1}{n^{1/3}}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^{1/3}}} = \frac{1}{1+0} = 1$

$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n}+1} = 1$ which is finite, so a_n converges to 1 by definition

c. $a_n = \frac{(n-2)!}{n!}$

$a_n = \frac{\cancel{(n-2)!}}{n(n-1)\cancel{(n-2)!}} = \frac{1}{n^2-n}$

$\lim_{n\to\infty} \frac{1}{n^2-n} = 0$

$\boxed{a_n \text{ converges to } 0 \text{ by definition}}$

d. $a_n = \dfrac{\cos \pi n}{n^2}$

e. $a_n = \dfrac{\ln \sqrt{n}}{n}$

5. Determine whether the sequence with the given nth term is monotonic. Discuss the boundedness of the sequence.

a. $a_n = \dfrac{n}{2^{n+2}}$

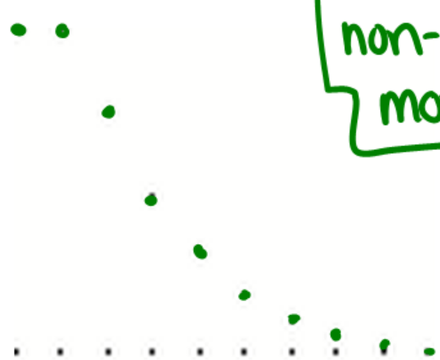
$a_1 = \dfrac{1}{2^{1+2}} = \dfrac{1}{8}$

$a_2 = \dfrac{2}{16} = \dfrac{1}{8}$

$a_3 = \dfrac{3}{32}$

$a_4 = \dfrac{4}{64} = \dfrac{1}{16}$

$a_1 \geq a_2 \geq a_3 \geq a_4$



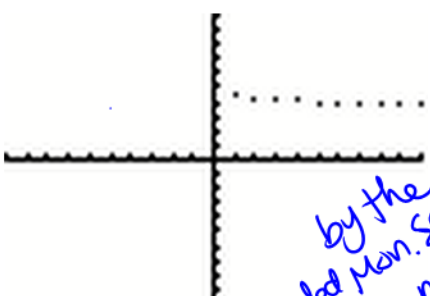
$\boxed{\text{non-increasing, thus monotonic}}$

b. $a_n = \left(-\frac{2}{3}\right)^n$

c. $a_n = ne^{-n/2}$

6. Use the Bounded Monotonic Sequences theorem to show that the sequence with the given nth term converges and use a graphing calculator to graph the first 10 terms of the sequence and find its limit.

$$a_n = 4 + \frac{1}{2^n}$$



① nonincreasing (see graph)

② Bounds

- Lower bound at 4 $\lim_{n\to\infty}\left(4+\frac{1}{2^n}\right) = 4+0 = 4$
- upper bound at 9/2

Since a_n is monotonic and bounded, a_n converges. $\lim_{n\to\infty}\left(4+\frac{1}{2^n}\right) = 4$

by the Bounded Mon. Seq. theorem