4/27/11

- Lecture 9.3

Friday

- Lecture 9.4
- Lecture 9.5

Monday

9.6

Next Wednesday

9.7

Theorem 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Steps for using the Integral Test.

1. Conditions.
    a. Sketch a graph or give some rationale for why the function f is positive and continuous for $x \geq 1$.
    b. Find the derivative of f and figure out at which integer greater than one f begins to be a decreasing function.
2. Evaluate $\int_N^{\infty} f(x)\,dx$, where N is the integer for which f begins to be a decreasing function. Most often, N is one.
3. State your conclusion <u>in words</u> stating the reason for convergence or divergence.

Step 3: Conclusion
$\sum_{n=1}^{\infty} \frac{2}{3n+5}$ diverges by the Integral test

1. Use the Integral Test to determine the convergence or divergence of the series.

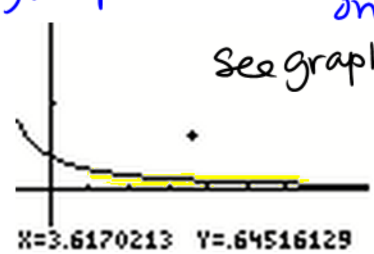
Let $f(x) = \dfrac{2}{3x+5}$

a. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n+5}$

Step 1: Conditions

a) f is positive and continuous on $[1,\infty)$

See graph



b) $f'(x) = -\dfrac{6}{(3x+5)^2} < 0$ for all x, so f is decreasing on $[1,\infty)$

Step 2: $\displaystyle\int_1^{\infty} \frac{2\,dx}{3x+5}$

$= \lim_{b\to\infty} \frac{1}{3}\int_1^b \frac{2\,dx \cdot 3}{3x+5}$

$= \lim_{b\to\infty} \frac{2}{3} \ln|3x+5| \Big|_1^b$

$= \frac{2}{3} \lim_{b\to\infty} \left(\ln|3b+5| - \ln|3\cdot 1+5|\right)$

$= \frac{2}{3}(\infty - \ln 8) = \infty$ diverges

b. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ Let $f(x)=\frac{\arctan x}{x^2+1}$

Step 1: Conditions

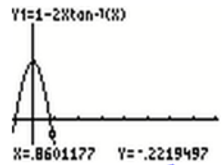
a) $\arctan x > 0$ and cont. on $[1,\infty)$ and $n^2+1>0$ and cont on $[1,\infty)$ so the quotient is also positive and continuous on $[1,\infty)$.

b) $f'(x)=\frac{\frac{1}{\cancel{x^2+1}}(\cancel{x^2+1}) - \arctan x\,(2x)}{(x^2+1)^2}$

$f'(x)=\frac{1-2x\arctan x}{(x^2+1)^2}$

$1-2x\arctan x<0$

consider $1-2x\arctan x=0$



$x^2+1>0$

$1-2x\arctan x<0$ on $[1,\infty)$

so f is decreasing on $[1,\infty)$

Step 2: Test

$\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx$

$=\frac{1}{2}\lim_{b\to\infty} (\arctan x)^2\Big|_1^b$

$=\frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2-\left(\frac{\pi}{4}\right)^2\right)$

so it converges since $\neq \pm\infty$

Step 3: Conclusion

$\sum_{n=1}^{\infty}\frac{\arctan n}{n^2+1}$ converges by the integral test.

c. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ Let $f(x)=\frac{\ln x}{x^3}$

Step 1: Conditions

a) $f>0$ on $[1,\infty)$ and also cont.

b) $f'(x)=\frac{\frac{x^{\cancel{3}2}}{\cancel{x}}-(\ln x)3x^2}{x^6}$

$f'(x)=\frac{\cancel{x^2}(1-3\ln x)}{x^{\cancel{6}4}}$

$f'(x)=\frac{1-3\ln x}{x^4}$

$0=1-3\ln x$

$\ln x=\frac{1}{3}$

$e^{1/3}=x$

$1.39\approx x$

f is decreasing on $[2,\infty)$

Step 2: Test

$\int_2^{\infty}\frac{\ln x}{x^3}\,dx=\lim_{b\to\infty}\int_2^b \frac{\ln x}{x^3}\,dx$

Consider $\int(\ln x)x^{-3}\,dx=-\frac{\ln x}{2x^2}+\frac{1}{2}\int\frac{dx}{x^3}$

$u=\ln x$ $\quad dv=x^{-3}dx$

$du=\frac{dx}{x}$ $\quad v=\frac{x^{-2}}{-2}$

$=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}$

$=-\frac{1}{2x^2}\left(\ln x+\frac{1}{2}\right)$

So $\lim_{b\to\infty}\left(-\frac{1}{2x^2}\left(\ln x+\frac{1}{2}\right)\right)$

$=\lim_{b\to\infty}-\frac{1}{2}\cdot\frac{\ln x}{x^2}-\lim_{b\to\infty}-\frac{1}{4x^2}$

$=-\frac{1}{2}\lim_{b\to\infty}\frac{\frac{1}{x}}{2x}=\frac{1}{2x^2}\,0$

$= -\frac{1}{2}(0)$

$= 0$

converges

Step 3: Conclusion

$\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ converges by the integral test.

Theorem 9.11 Convergence of *p*-Series

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

1. Converges if $p > 1$, and
2. Diverges if $p \leq 1$.

2. Determine the convergence or divergence of the *p*-series.

a. $\sum_{n=1}^{\infty} \frac{3}{\sqrt[5]{n^3}} = 3\sum_{n=1}^{\infty} \frac{1}{n^{3/5}}$

$\boxed{p = \frac{3}{5} \leq 1, \text{ so } \sum_{n=1}^{\infty} \frac{3}{\sqrt[5]{n^3}} \text{ is a divergent p-series.}}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

$\boxed{p = \pi > 1, \text{ so } \sum_{n=1}^{\infty} \frac{1}{n^{\pi}} \text{ is a convergent p-series.}}$

c. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\boxed{p = 2 > 1, \text{ so } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a convergent p-series.}}$

3. Find the positive values of p for which the series converges.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

b. $\displaystyle\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$