Theorem 9.12 Direct Comparison Test

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Steps for using the Direct Comparison Test.

1. Decide if you think the series will converge or diverge. Then use a **similar series** for which it is easy to establish convergence or divergence which has terms which are
   a. **Greater** for all n in a term by term comparison with the given series if you want to prove **convergence** and
   b. **Lesser** for all n in a term by term comparison with the given series if you want to prove **divergence**.
2. Show that the **similar series** converges or diverges.
3. State your conclusion **in words** stating the reason for convergence or divergence.

1. Use the Direct Comparison Test to determine the convergence or divergence of the series.

   a. $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

nth term test:
$\lim_{n\to\infty} \frac{1}{3n^2+2} = 0$
so more work!

① choose $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ as comparison series

conditions:
need to show that $\frac{1}{3n^2} > \frac{1}{3n^2+2}$

$n=1$: $\frac{1}{3} \overset{?}{>} \frac{1}{5}$ yes

$n=2$: $\frac{1}{12} \overset{?}{>} \frac{1}{14}$ yes

② $\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series $[p=2>1]$

③ $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges by the DCT.

b. $\sum_{n=1}^{\infty}\frac{4^n}{3^n-1}$

(1) Choose $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ as comparison series

conditions: ~~need to~~ show that $\left(\frac{4}{3}\right)^n < \frac{4^n}{3^n-1}$

$n=1$: $\frac{4}{3} \overset{?}{<} \frac{4}{2}$ yes

$n=2$: $\frac{16}{9} \overset{?}{<} \frac{16}{8}$ yes

(2) $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ is a divergent geometric series $\left[r=\frac{4}{3}\geq 1\right]$.

(3) $\sum_{n=1}^{\infty}\frac{4^n}{3^n-1}$ diverges by the DCT.

c. $\sum_{n=1}^{\infty}\frac{1}{4\sqrt[3]{n}-1}$

nth term test for div.: $\lim_{n\to\infty}\frac{1}{4n^{1/3}-1}=0$ so more work!

(1) Choose $\sum_{n=1}^{\infty}\frac{1}{4n^{1/3}}$ as comparison series

conditions:

Show $\frac{1}{4n^{1/3}} < \frac{1}{4n^{1/3}-1}$

$n=1$: $\frac{1}{4} \overset{?}{<} \frac{1}{3}$ yes

$n=8$ $\frac{1}{8} \overset{?}{<} \frac{1}{7}$ yes

(2) $\sum_{n=1}^{\infty}\frac{1}{4n^{1/3}}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ is a divergent p-series $\left[p=\frac{1}{3}\leq 1\right]$

(3) $\sum_{n=1}^{\infty}\frac{1}{4\sqrt[3]{n}-1}$ diverges by the DCT

Theorem 9.12 Limit Comparison Test

$a_n > 0,\ b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is finite and positive. Then the two series either both converge or both diverge.

Steps for using the Limit Comparison Test.

1. Decide if you think the series will converge or diverge. Then use a **similar series** for which it is easy to establish convergence or divergence.
2. Evaluate the limit of the ratio of the **sequence** in the given series to the **sequence** in the similar series.
3. State your conclusion **in words** stating the reason for convergence or divergence.

2. Use the Limit Comparison Test to determine the convergence or divergence series.

nth term test for div: $\lim_{n\to\infty} \frac{2}{3^n-1} = 0$ so more work!

a. $\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$

① Choose $\sum_{n=1}^{\infty} \frac{2}{3^n}$ as comparison series

$\frac{2}{3^n} > 0$, for $n \geq 1$

$\frac{2}{3^n-1} > 0$ for $n \geq 1$

condition

② a) $\lim_{n\to\infty} \frac{\frac{2}{3^n}}{\frac{2}{3^n-1}} = \lim_{n\to\infty} \frac{\frac{3^n - 1}{3^n}}{\frac{3^n}{3^n}}$

$= \lim_{n\to\infty} \frac{1 - \frac{1}{3^n}}{1}$

$= 1$, which is finite and positive

b) $\sum_{n=1}^{\infty} \frac{2}{3^n} = 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series $[|r| = \frac{1}{3},\ 0 < \frac{1}{3} < 1]$

③ $\sum_{n=1}^{\infty} \frac{2}{3^n-1}$ converges by the LCT

b. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)} = \sum_{n=1}^{\infty}\frac{1}{n^3-n}$

1. Choose $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3}$ as comparison series

$$\left.\begin{aligned}&\frac{1}{n^3} > 0,\ n \geq 2\\ &\frac{1}{n^3-n} > 0,\ n\geq 2\end{aligned}\right\}\ \text{conditions}$$

2 a) $$\lim_{n\to\infty}\frac{\frac{1}{n^3}}{\frac{1}{n^3-n}} = \lim_{n\to\infty}\frac{\frac{n^3}{n^3}-\frac{n}{n^3}}{\frac{n^3}{n^3}}$$

$$= \lim_{n\to\infty}\frac{1-\frac{1}{n^2}}{1}$$

$= 1$, finite and positive

b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3}$ is convergent p-series $[p=3>1]$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}$ converges by the LCT.

c. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2-1}{3n^5+2n+1}$

1. Choose $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2}{3n^5} = \sum_{n=1}^{\infty}\frac{2}{3n^3}$ as comparison series

$$\left.\begin{aligned}&\frac{2}{3n^3} > 0,\ n \geq 1\\ &\frac{2n^2-1}{3n^5+2n+1} > 0,\ n\geq 1\end{aligned}\right\}\ \text{conditions}$$

2 a) $$\lim_{n\to\infty}\frac{\frac{2}{3n^3}}{\frac{2n^2-1}{3n^5+2n+1}} = \lim_{n\to\infty}\frac{2(3n^5+2n+1)}{3n^3(2n^2-1)}$$

$$= \lim_{n\to\infty}\frac{6n^5+4n+2}{6n^5-3n^3}$$

$= 1$, finite and positive

b) $\displaystyle\sum_{n=1}^{\infty}\frac{2}{3n^3} = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series $[p=3>1]$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2-1}{3n^5+2n+1}$ converges by the LCT