4/6/11

- Finish 10.5
- Review 10.1-10.5

Friday

- Exam 4/10.1-10.5 → I will provide the front cover formulas (book) front and back
- HW 10.1-10.5 due → you may write steps and formulas ↳ no sample problems

Monday

Start Ch. 9

*review sequences and series before class!

Standard Equation of a Parabola with vertex (h,k) and directrix $y=k-p$ is

$(x-h)^2=4p(y-k)$. Vertical axis of symmetry

Standard Equation of a Parabola with vertex (h,k) and directrix $x=h-p$ is

$(y-k)^2=4p(x-h)$. Horizontal axis of symmetry

The focus lies on the axis p units (directed distance) from the vertex. The coordinates of the focus are as follows:

$(h,k+p)$ Vertical axis of symmetry

$(h+p,k)$ Horizontal axis of symmetry

A **focal chord** is a line segment which passes through the focus of a parabola and has endpoints on the parabola.

The specific focal chord perpendicular to the axis of the parabola is the **latus rectum**.

A surface is considered **reflective** if the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the **angle of incidence** and the angle corresponding to the outgoing ray is the **angle of reflection**.

Let P be a point on a parabola. The tangent line makes equal angles with the following two lines:

1. The line passing through P and the focus.
2. The line passing through P parallel to the axis of the parabola.

1. Consider the parabola $y^2 - 4x - 4 = 0$
   a. Find the vertex, focus, and directrix of the parabola and sketch its graph.
   b. Find dy/dx at x = 1.

2. Find an equation of the parabola with

   directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

Standard Equation of an Ellipse with center (h,k) and major and minor axes of length $2a$ *and* $2b$ where $a>b,$ is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Major axis is horizontal

Or

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

Major axis is vertical

The foci lies on the **major** axis, c units from the center, with $c^2=a^2-b^2$.

Reflective Property of an Ellipse: Let P be a point on an ellipse. The tangent line to an ellipse at point P makes equal angles with the lines through P and the foci.

Eccentricity of an Ellipse: The eccentricity e of an ellipse is given by

$$e=\frac{c}{a}.$$

This measures the ovalness of an ellipse. When the eccentricity is very small and the foci are close to the center, the ellipse is nearly circular. When the eccentricity is close to one and the foci are close to the vertices, the ellipple will be elongated.

3. Find the equation of the ellipse with

$$\text{eccentricity: } \frac{1}{2} \text{ and vertices: } (0,2) \text{ and } (8,2).$$

4. Consider the ellipse $x^2+2y^2+8x+4y=1$.

a. Find the center, foci, and vertices of the ellipse, and sketch its graph.

b. Find the equation(s) of the

i. Tangent line(s) at y = 2

ii. Normal lines at x = 2

Standard Equation of a Hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Transverse axis is horizontal

Or

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Transverse axis is vertical

The vertices are a units from the center. The foci are c units from the center, with $c^2=a^2+b^2$.

<u>**Asymptotes of a Hyperbola**</u>:

For a <u>**horizontal**</u> transverse axis, the equations of the asymptotes are

$$y=k+\frac{b}{a}(x-h) \quad \text{and} \quad y=k-\frac{b}{a}(x-h).$$

For a <u>**vertical**</u> transverse axis, the equations of the asymptotes are

$$y=k+\frac{a}{b}(x-h) \quad \text{and} \quad y=k-\frac{a}{b}(x-h).$$

The line segment of length $2b$ joining the points which are b units away from the center is referred to as the <u>**conjugate axis**</u>. The **eccentricity** ***e*** is $e={}^{c}\!/_{a}$.

5. Find the equation of the hyperbola with

$$\text{asymptotes: } y=\pm\frac{3}{4}x \text{ and focus: } (10,0).$$

6. Consider the hyperbola

   a. Find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.

$$\frac{x^2}{25}-\frac{y^2}{9}=1$$

   b. Find equations for the

      i. Tangent lines at x = 10.

      ii. Normal lines at x = 10.

7. Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

   a. $12x^2 + 2x + 2y^2 = 15$

   b. $x^2 - x + y^2 - 5y = 1 + x^2$

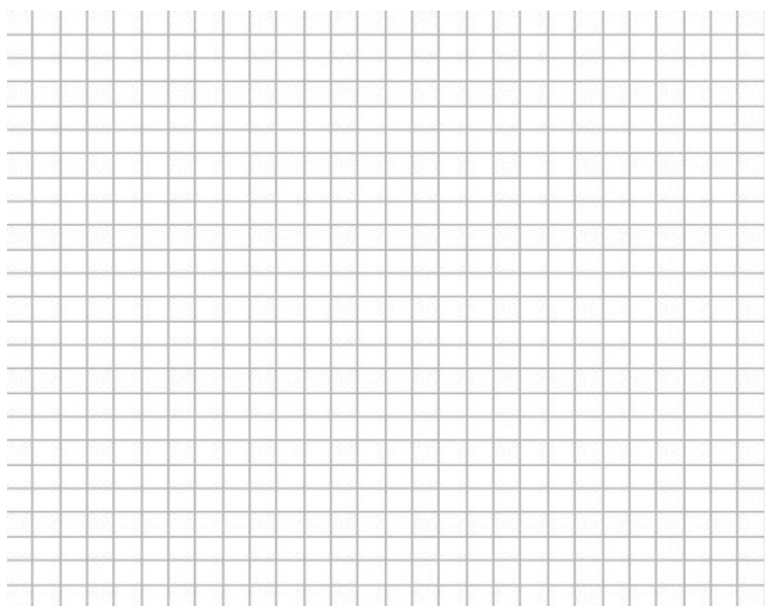
   c. $-x^2 - 2x + 3y^2 = y + 12$

   d. $-x^2 = y^2 - 1$

   e. $2x^2 - 8x + y^2 + 5y = 6$

8. Consider the parametric equation $x = \tan^2\theta$ and $y = \sec^2\theta$.

   a. Eliminate the parameter and graph the parametric equation by hand, indicating the orientation.



   b. Evaluate $\dfrac{dy}{dx}$.

PARAMETRIC FORM OF THE DERIVATIVE

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope at C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

HIGHER ORDER DERIVATIVES OF PARAMETRIC EQUATIONS

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope at C at (x, y) is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

In general we have,

$$\frac{d^ny}{dx^n} = \frac{\frac{d}{dt}\left[\frac{d^{n-1}y}{dx^{n-1}}\right]}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$

9. Consider the parametric equations, $x=\cos\theta$ and $y=2\sin\theta$ at $\theta=\dfrac{\pi}{4}$.

a. Find $\dfrac{dy}{dx}$.

b. Find $\dfrac{d^2y}{dx^2}$.

c. Find the slope and concavity at the given value of the parameter.

10. Consider the curve $x = t + 1, \quad y = t^2 + 3t$.

a. Find all points of horizontal tangency to the curve.

b. Find all points of vertical tangency to the curve

ARC LENGTH IN PARAMETRIC FORM

If a smooth curve *C* is given by the equations $x = f(t)$ and $y = g(t)$, such that *C* does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of *C* over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt.$$

NOTE: When applying the arc length formula to a curve, be sure that the curve is traced only once on the interval of integration.

11. Find the arc length of the curve $x = t, \quad y = \frac{t^5}{10} + \frac{1}{6t^3}$ on the interval $1 \le t \le 2.$

AREA OF A SURFACE OF REVOLUTION

If a smooth curve *C* is given by the equations $x = f(t)$ and $y = g(t)$, does not cross itself on the interval $a \le t \le b$, then the area *S* of the surface of revolution formed by revolving *C* about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $x-$axis: $g(t) \ge 0$

2. $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the $y-$axis: $f(t) \ge 0$

12. Find the area of the surface generated by revolving the curve about the y- axis.

$$x = 4\cos\theta \text{ and } y = 4\sin\theta \text{ on the interval } 0 \le \theta \le \frac{\pi}{2}$$

SLOPE IN POLAR FORM

If f is a differentiable function of θ, then the **slope** of the **tangent line to the graph** of $r = f(\theta)$ at the point (r,θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}, \qquad \frac{dx}{d\theta} \ne 0 \text{ at } (r,\theta).$$

TANGENT LINES AT THE POLE

If $f(\alpha) = 0$ and $f'(\alpha) \ne 0$, then the line $\theta = \alpha$ is **tangent at the pole to the graph of** $r = f(\theta)$.

AREA IN POLAR COORDINATES

If f is continuous and **nonnegative** on the interval $[\alpha,\beta],\ 0<\beta-\alpha\le 2\pi,$ then the area of the region bounded by the graph of $r=f(\theta)$ between the radial lines of $\theta=\alpha$ and $\theta=\beta$ is given by

$$A=\frac{1}{2}\int_\alpha^\beta\left[f(\theta)\right]^2 d\theta$$
$$=\frac{1}{2}\int_\alpha^\beta r^2 d\theta.$$

ARC LENGTH IN POLAR FORM

Let f be a function whose derivative is continuous on an interval $\alpha\le\theta\le\beta$. The **length of the graph** of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$ is

$$s=\int_\alpha^\beta\sqrt{\left[f(\theta)\right]^2+\left[f'(\theta)\right]^2}\,d\theta=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

AREA OF A SURFACE OF REVOLUTION

Let f be a function whose derivative is continuous on an interval $\alpha\le\theta\le\beta$. The **area of the surface** formed by revolving the graph of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$ about the indicated line is as follows.

1. $S=2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{\left[f(\theta)\right]^2+\left[f'(\theta)\right]^2}\,d\theta$ (about the polar axis)

2. $S=2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{\left[f(\theta)\right]^2+\left[f'(\theta)\right]^2}\,d\theta$ $\left(\text{about the line } \theta=\frac{\pi}{2}\right)$

13. Find two sets of polar coordinates for the rectangular coordinate $\left(3,-\sqrt{3}\right)$.

QIV

need (r,θ), we have (x,y)

$r=\sqrt{x^2+y^2}$

$r=\sqrt{(3)^2+(-\sqrt{3})^2}$

$r=\sqrt{12}$

$r=2\sqrt{3}$

$\tan\theta=-\frac{\sqrt{3}}{3}$

$\tan\theta=-\frac{1}{\sqrt{3}}$

$\tan\theta=\frac{-1/2}{\sqrt{3}/2}$

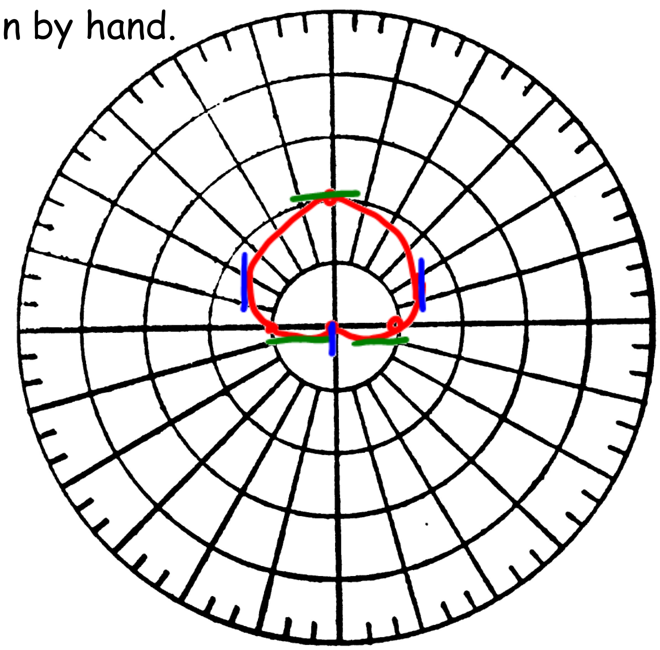
$\theta=\frac{11\pi}{6}$

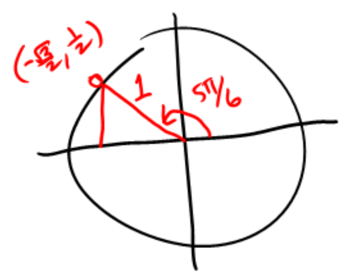
$$\boxed{\begin{array}{c}\left(2\sqrt{3},\frac{11\pi}{6}\right)\\ \left(-2\sqrt{3},\frac{5\pi}{6}\right)\end{array}}$$

14. Consider the polar equation $r=1+\sin\theta$. → limaçon, cardioid since $a=b=1$ and $\frac{1}{1}=1$

a. Sketch a graph of the polar equation by hand.

r	θ
1	0
2	$\pi/2$
1	π
0	$\frac{3\pi}{2}$
$\frac{3}{2}$	$\frac{\pi}{6}$
$\frac{3}{2}$	$\frac{5\pi}{6}$





b. Find all points of horizontal and vertical tangency.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$$

$r = 1+\sin\theta$

$r' = \cos\theta$

$$\frac{dy}{dx} = \frac{(1+\sin\theta)\cos\theta + \cos\theta\sin\theta}{-(1+\sin\theta)\sin\theta + \cos\theta\cos\theta}$$

$$\frac{dy}{dx} = \frac{\cos\theta(1+\sin\theta+\sin\theta)}{-\sin\theta-\sin^2\theta+\cos^2\theta}$$

Horizontal tangency

$0 = \cos\theta(1+2\sin\theta)$

$\cos\theta = 0$ or $1+2\sin\theta = 0$

$\theta = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}$

or $\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\boxed{(2, \pi/2), \left(\frac{1}{2}, 7\pi/6\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)}$

Vertical tangency

$0 = -\sin\theta - \sin^2\theta + \cos^2\theta$

$0 = \sin\theta + \sin^2\theta - (1-\sin^2\theta)$

$0 = \sin\theta + 2\sin^2\theta - 1$

$0 = (2\sin\theta - 1)(\sin\theta + 1)$

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = -1$

$\theta = \pi/6, 5\pi/6$ $\qquad$ $\theta = \frac{3\pi}{2}$

$\boxed{\left(\frac{3}{2}, \pi/6\right), \left(\frac{3}{2}, 5\pi/6\right), (0, 3\pi/2)}$

c. Find the tangents at the pole.

$r = 1+\sin\theta$

$0 = 1+\sin\theta$

$\theta = 3\pi/2$

at $\theta = \frac{3\pi}{2}$

$\frac{dr}{d\theta} = \cos\frac{3\pi}{2}$

$\frac{dr}{d\theta} = 0$

$\boxed{\text{No tangents at the pole.}}$

d. Find the area of the interior.

$$A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1+\sin\theta)^2\,d\theta\right]$$

$$A = \int_{-\pi/2}^{\pi/2}(1+2\sin\theta+\sin^2\theta)\,d\theta$$

e. Find the arc length of the curve over the interval $0 \le \theta \le 2\pi$.

$$S = 2\int_{-\pi/2}^{\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$S = 2\int_{-\pi/2}^{\pi/2} \sqrt{1+2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}}\, d\theta$$

$$S = 2\int_{-\pi/2}^{\pi/2} \sqrt{2(\sin\theta+1)}\, d\theta$$

$$S = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} \sqrt{1+\sin\theta}\ \frac{\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$S = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{\sqrt{\cos^2\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$S = 2\sqrt{2}\int_{-\pi/2}^{\pi/2} -\cos\theta\,(1-\sin\theta)^{-1/2}\, d\theta$$

$$S = -2\sqrt{2}\left(2(1-\sin\theta)^{1/2}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$S = -4\sqrt{2}\left[(1-1)^{1/2} - (1-(-1))^{1/2}\right]_{-\pi/2}^{\pi/2}$$

$$S = -4\sqrt{2}\,(-\sqrt{2})$$

$$\boxed{S = 8 \text{ units}}$$

$$\cos^2 A = \frac{1+\cos 2A}{2}$$

$$2\cos^2 A = 1+\cos 2A$$

If you end up with $\sqrt{\cos\theta+1}$, rewrite as $\sqrt{2\cos^2\frac{\theta}{2}} = \sqrt{2}\cos\frac{\theta}{2}$

*This occurs if you have $r = 1-\cos\theta$

f. Find the area of the surface formed by revolving the curve about the polar axis over the interval $0 \le \theta \le 2\pi$.

$$S = 2\pi\int_0^{2\pi} (1+\sin\theta)\sin\theta\sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$S = 2\pi\int_0^{2\pi} \underline{(1+\sin\theta)}\sin\theta\left(2\underline{(1+\sin\theta)}\right)^{1/2} d\theta$$

$$S = 2\sqrt{2}\pi\int_0^{2\pi} \sin\theta\,(1+\sin\theta)^{3/2}\, d\theta$$

$$S = 2\sqrt{2}\pi\int_0^{2\pi} \left(\sin^2\theta\,(1+\sin\theta)^3\right)^{1/2}$$

$$\sin\theta = \sqrt{\sin^2\theta}$$