Definition of Power series

If x is a variable, then the infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_n x^n + \cdots$$ is called a **power series**.

More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_n(x-c)^n + \cdots$$

is called a **power series centered at c**, where c is a constant.

1. State where the power series is centered.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$

$a_n = \frac{1}{n!}$

$(x-(-2))^n$

$\boxed{\text{centered at } -2}$

b. $\displaystyle\sum_{n=2}^{\infty} \frac{(n-5)^n x^n}{\ln n}$

$a_n = \frac{(n-5)^n}{\ln n}$

$(x)^n$

$\boxed{\text{Centered at } 0}$

Convergence of a Power Series

For a power series centered at c, precisely **one** of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$. If the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the series converges is the **interval of convergence** of the power series.

2. Find the radius of convergence of the power series.

a. $\displaystyle\sum_{n=0}^{\infty}(2x)^n$

Geometric, so $|2x| = r$ and when $|r|<1$ the series converges.

$|2x|<1 \rightarrow |2|\cdot|x|<1 \rightarrow |x|<\frac{1}{2} = R$

$\boxed{\text{radius of convergence is } \frac{1}{2}}$

b. $\displaystyle\sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{n!}$

$$\left|\frac{\overset{(2n+2)!}{(2(n+1))!}\,x^{\overset{2n+2}{2(n+1)}}}{(n+1)!}\cdot\frac{n!}{(2n)!\,x^{2n}}\right|$$

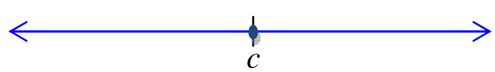
$$\left|\frac{\overset{2(\cancel{n+1})}{(2n+2)}(2n+1)\cancel{(2n)!}\,x^{2n+2-2n}\,\cancel{n!}}{\cancel{(n+1)}\cancel{(n!)}\,\cancel{(2n)!}}\right| = |2(2n+1)x^2| \quad \text{so } \lim_{n\to\infty}|2(2n+1)x^2| = \infty$$

so $x = 0$ for convergence and radius of convergence is zero.

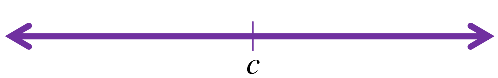
Endpoint Convergence

Note that for a power series whose radius of convergence is a finite number *R*, each endpoint must be tested separately for convergence or divergence. The interval of convergence of a power series can take any one of the six forms below.

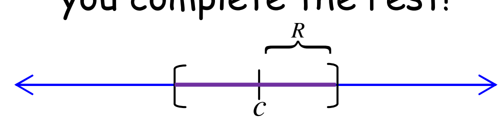
Radius: 0

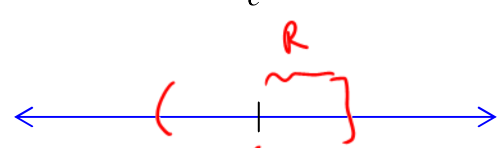


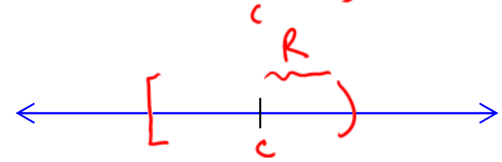
Radius: Infinity

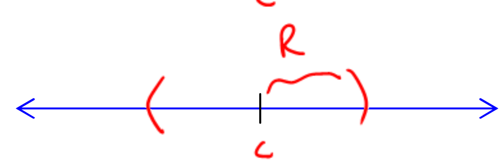


Radius: *R*—I showed you one possibility, now you complete the rest!









3. Find the interval of convergence of the power series. Be sure to include a check for endpoint convergence at the endpoints of the interval.

a. $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

b. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)(n+2)}$

$|x|<p$

$-p<x<p$

c. $\sum_{n=1}^{\infty}\frac{(x-3)^{n-1}}{3^{n-1}}$

$\left|\frac{(x-3)^{(n+1)-1}}{3^{(n+1)-1}}\cdot\frac{3^{n-1}}{(x-3)^{n-1}}\right| = \left|(x-3)^{n-(n-1)}\cdot 3^{n-1-n}\right|$

$= \left|\frac{(x-3)^1}{3}\right| < 1$ So... $-1<\frac{x-3}{3}<1$

$-3<x-3<3$

interval of convergence: $0<x<6$

Endpoints:

At $x=0$

$$\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{3^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cancel{3^{n-1}}}{\cancel{3^{n-1}}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(-1)}$$

$$= -\sum_{n=1}^{\infty} (-1)^n$$

$\rightarrow$ Divergent geometric series $[|r| = |-1| = 1 \geq 1]$

At $x=6$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 1$$

nth term test:

$$\lim_{n\to\infty} 1 = 1 \neq 0$$

Diverges by the nth term test

$$\boxed{\text{So... } \sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}} \text{ converges on } (0,6)}$$