MATH 251/GRACEY NAME Key

PRACTICE/CHAPTER 8

1. Find the integral.

a. $\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C}$

$u = \ln x$
$du = \frac{dx}{x}$

b. $\int \ln\sqrt{x^2-1}\,dx = x\ln\sqrt{x^2-1} - \int \frac{x \cdot x\,dx}{x^2-1}$

$u = \ln\sqrt{x^2-1}$
$u = \frac{1}{2}\ln(x^2-1)$
$du = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{x^2-1}dx$
$du = \frac{x\,dx}{x^2-1}$

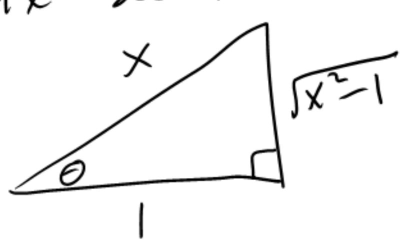
$\int dv = \int dx$
$v = x$

$= x\ln\sqrt{x^2-1} - \int \frac{x^2 dx}{x^2-1}$

$= x\ln\sqrt{x^2-1} - \int dx - \int \frac{dx}{x^2-1}$

$$\begin{array}{r} 1 + \frac{1}{x^2-1} \\ x^2-1 \overline{)\, x^2 + 0x + 0} \\ -(x^2 \quad -1) \\ \hline 1 \end{array}$$

$x = 1\sec\theta$, $x^2 - 1 = \tan^2\theta$
$dx = \sec\theta\tan\theta\,d\theta$



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$\int \frac{\sec\theta\cancel{\tan\theta}\,d\theta}{\tan^{\cancel{2}}\theta} = \int \csc\theta\,d\theta$

$\frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta}$

$$\int \csc\theta\, d\theta = -\ln\left|\csc\theta + \cot\theta\right|$$

$$= -\ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right|$$

$$= -\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right|$$

$$= \boxed{x \ln\sqrt{x^2-1} - x + \ln\left|\frac{x+1}{\sqrt{x^2-1}}\right| + C}$$

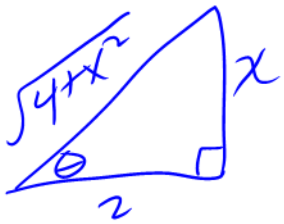
c. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{(8\tan^3\theta)(\cancel{2}\sec^{\cancel{2}}\theta\,d\theta)}{(\cancel{2}\cancel{\sec\theta})}$

$x = 2\tan\theta$

$\sqrt{4+x^2} = 2\sec\theta$

$dx = 2\sec^2\theta\,d\theta$

$x^3 = 8\tan^3\theta$



$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$= 8\int \tan^2\theta\,\sec\theta\tan\theta\,d\theta$

$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$

$= 8\int \sec^2\theta\,\sec\theta\tan\theta\,d\theta - 8\int \sec\theta\tan\theta\,d\theta$

$u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta$

$= 8\int u^2\,du - 8\sec\theta + C$

$= \frac{\sec^3\theta}{3} - \sec\theta + C = 8\left[\frac{(\sqrt{4+x^2})^3}{24} - \frac{\sqrt{4+x^2}}{2} + C\right]$

d. $\int \frac{1}{1-\cos\theta}\,d\theta = \int \csc^2\theta\,d\theta + \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\cot\theta + \int u^{-2}\,du$

$\frac{1}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} = \frac{1+\cos\theta}{1-\cos^2\theta}$

$= \frac{1+\cos\theta}{\sin^2\theta}$

$= \frac{1}{\sin^2\theta} + \frac{\cos\theta}{\sin^2\theta}$

$= \csc^2\theta + \frac{\cos\theta}{\sin^2\theta}$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$= -\cot\theta - \frac{1}{\sin\theta} + C$

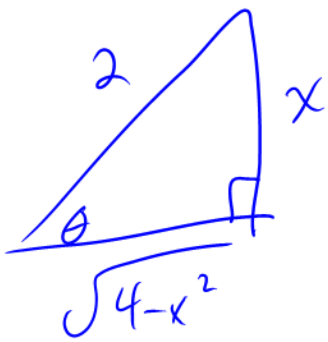
$= \boxed{-\cot\theta - \csc\theta + C}$

e. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = +12 \int \frac{(\cancel{+2}\cancel{\cos\theta}\,d\theta)}{4\sin^2\theta\,(\cancel{2\cos\theta})}$

$x = 2\sin\theta$, $dx = -2\cos\theta\,d\theta$

$x^2 = 4\sin^2\theta$, $\sqrt{4-x^2} = 2\cos\theta$

$= 3\int \frac{d\theta}{\sin^2\theta}$



$= 3\int \csc^2\theta\,d\theta$

$= -3\cot\theta + C$

$= \boxed{\frac{-3\sqrt{4-x^2}}{x} + C}$

f. $\int \frac{x-28}{x^2-x-6}\,dx = -5\int\frac{dx}{x-3} + 6\int\frac{dx}{x+2} = -5\ln|x-3| + 6\ln|x+2| + c$

$= \boxed{\ln\left|\frac{(x+2)^6}{(x-5)^5}\right| + C}$

Partial Fraction Decomp.

$\frac{x-28}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = -\frac{5}{x-3} + \frac{6}{x+2}$

$x - 28 = A(x+2) + B(x-3)$

$x - 28 = Ax + 2A + Bx - 3B$

$1x - 28 = (A+B)x + (2A-3B)$

$A + B = 1 \rightarrow A = 1 - B = 1 - 6 = -5$

$2A - 3B = -28$

$2(1-B) - 3B = -28 \rightarrow 2 - 2B - 3B = -28$

$30 = 5B$

$B = 6$

2. Evaluate the definite integral.

a. $\int_0^{\pi/4} \tan^3 x dx = \int_0^{\pi/4} \tan^2 x \tan x dx$

$= \int_0^{\pi/4} (\sec^2 x - 1) \tan x dx$

$= \int_0^{\pi/4} \sec^2 x \tan x dx - \int_0^{\pi/4} \tan x dx$

$u = \sec x$

$du = \sec x \tan x dx$

$= \int_0^{\pi/4} \sec x \sec x \tan x dx - \left(-\ln|\cos x|\right)\Big|_0^{\pi/4}$

$= \frac{\sec^2 x}{2}\Big|_0^{\pi/4} + \ln|\cos x|\Big|_0^{\pi/4}$

$= \frac{(\sqrt{2})^2 - (1)^2}{2} + \ln(\sqrt{2}/2) - \ln 1 = \boxed{\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right)}$

b. $\int_0^2 e^{-x} \cos x dx = -e^{-x} \cos x - \int_0^2 e^{-x} \sin x dx$

$dv = e^{-x} dx$

$v = -e^{-x}$

$u = \cos x$

$du = -\sin x dx$

$= -e^{-x} \cos x - \left[-e^{-x} \sin x + \int_0^2 e^{-x} \cos x dx\right]$

so we have...

$\int_0^2 e^{-x} \cos x dx = -e^{-x} \cos x + e^{-x} \sin x - \int_0^2 e^{-x} \cos x dx$

$dv = e^{-x} dx$

$v = -e^{-x}$

$u = \sin x$

$du = \cos x dx$

$2\int_0^2 e^{-x} \cos x dx = \left(-e^{-x} \cos x + e^{-x} \sin x\right)\Big|_0^2$

$\int_0^2 e^{-x} \cos x dx = \left[\frac{e^{-x}}{2}(2\sin x - \cos x)\right]\Big|_0^2$

$= \left(\frac{e^{-2}}{2}(2\sin 2 - \cos 2)\right) - \left[1(0-1)\right]$

$$= \boxed{\frac{e^{-2}}{2}\left(2\sin 2 - \cos 2\right) + 1}$$

c. $\int_0^e \ln x^2 dx$

$\ln x^2 = 2\ln x$

$u = \ln x \qquad dv = dx$

$du = \frac{dx}{x} \qquad v = x$

$$= 2\int_0^e \ln x\,dx = 2\lim_{a\to 0^+}\left[x\ln x - \int_a^e \frac{x\,dx}{x}\right]$$

$$= 2\lim_{a\to 0^+}\left[x\ln x - \int_a^e dx\right]$$

$$= 2\lim_{a\to 0^+}\left[x\ln x - x\right]_a^e$$

$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty}$ $\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

L'Hopital

$$= 2\left[(e\ln e - e) - (a\ln a - a)\right]$$

$$= 2(0 - 0) = \boxed{0}$$

d. $\int_1^\infty \frac{1}{x^2+5}dx$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+5} = \lim_{b\to\infty} \frac{1}{\sqrt{5}} \arctan \frac{x}{\sqrt{5}}\Big|_1^b$$

$$= \boxed{\frac{1}{\sqrt{5}}\left(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{5}}\right)}$$

3. For the following limits:

a. Describe the type of indeterminate form (if any) that is obtained by direct substitution.

b. Evaluate the limit, using L'Hôpital's Rule if necessary.

i. $\lim_{x\to\infty} x \ln x$ $= \infty$

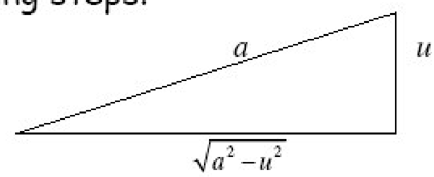
b) ii. $\lim_{x\to 0} \frac{\arcsin x}{x}$ $= \frac{0}{0}$ a) $\rightarrow \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = \boxed{1}$

$$\frac{\frac{\partial}{\partial x}\arcsin x}{\frac{\partial}{\partial x} x} = \frac{\frac{1}{\sqrt{1-x^2}}}{1} = \frac{1}{\sqrt{1-x^2}}$$

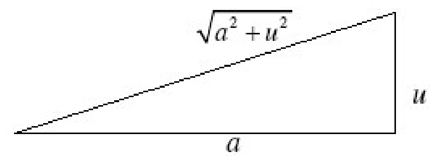
TRIGONOMETRIC SUBSTITUTION ($a > 0$)

Let $f(c) = 0$ where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

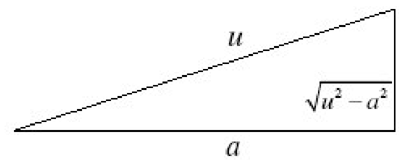
1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$.



Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.

Indeterminate Forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty - \infty,\ 0 \cdot \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$
2. If f is continuous on the interval $(-\infty,b]$, then
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$
3. If f is continuous on the interval $(-\infty,\infty)$, then
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx,\ c \text{ is any real number.}$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at *b*, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$
2. If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at *a*, then
$$\int_c^b f(x)\,dx = \lim_{c\to a^+}\int_a^c f(x)\,dx$$
3. If f is continuous on the interval $[a,b]$, except for some *c* in (a,b) at which *f* has an infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

In the first two cases, the improper integral **converges** if the limit exists—otherwise the improper integral **diverges**. In the third case, the improper integral on the left diverges if **either** of the improper integrals on the right diverges.