When you are done with your homework you should be able to…

- π Use a double integral to represent the volume of a solid region
- π Use properties of double integrals
- π Evaluate a double integral as an iterated integral

Warm-up: Evaluate the iterated integral $\int_0^{\pi}\int_0^{\pi/2}\sin^2 x\cos^2 y\,dy\,dx$.

$$\int_0^{\pi}\sin^2 x\left[\int_0^{\pi/2}\left(\frac{1+\cos 2y}{2}\right)dy\right]dx = \int_0^{\pi}\sin^2 x\left[\frac{y}{2}+\frac{\sin 2y}{4}\Big|_0^{\pi/2}\right]dx$$

$$=\frac{\pi}{4}\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$$

$$=\frac{\pi}{4}\left[\frac{x}{2}-\frac{\sin 2x}{4}\Big|_0^{\pi}\right]$$

$$=\frac{\pi}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi^2}{8}}$$

$$\int_0^4\int_0^4 f(x,y)\,dy\,dx$$

ACTIVITY: The table below shows values of a function f over a square region R. Divide the region into 16 equal squares and select (x_i, y_i) to be the point in the ith square closest to the origin. Compare this approximation with that obtained by using the point in the ith square furthest from the origin.

$f(0,0)=32 \rightarrow (0,0,32)$

$f(4,4)=0 \rightarrow (4,4,0)$

x \ y	0	1	2	3	4
0	32	31	28	23	16
1	31	30	27	22	15
2	28	27	24	19	12
3	23	22	19	14	7
4	16	15	12	7	0

$$\int_0^4\int_0^4 f(x,y)\,dy\,dx \approx (32+31+28+23)+(31+30+27+22)+(28+27+24+19)+(23+22+19+14)$$

$$=\boxed{400}$$

$$\int_0^4\int_0^4 f(x,y)\,dy\,dx \approx (30+27+22+15)+(27+24+19+12)+(22+19+14+7)+(15+12+7+0)$$

$$=\boxed{272}$$

DEFINITION: DOUBLE INTEGRAL

If f is defined on a closed, bounded region R in the *xy*-plane, then the **double integral of *f* over *R*** is given by

$$\int_R\int f(x,y)\,dA = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i,y_i)\,\Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

VOLUME OF A SOLID REGION

If f is integrable over a plane region R and $f(x,y)\ge 0$ for all (x,y) in R, then the volume of the solid region that lies above R and below the graph of f is defined as

$$V = \int_R\int f(x,y)\,dA$$

Example 1: Sketch the region R and evaluate the iterated integral $\int_R\int f(x,y)\,dA$.

$$\int_0^4\int_{\frac{1}{2}y}^{\sqrt{y}} x^2y^2\,dx\,dy$$

$$\int_0^4\int_{\frac{1}{2}y}^{\sqrt{y}} x^2y^2\,dx\,dy = \int_0^4 y^2\frac{x^3}{3}\Big|_{\frac{y}{2}}^{\sqrt{y}}\,dy$$

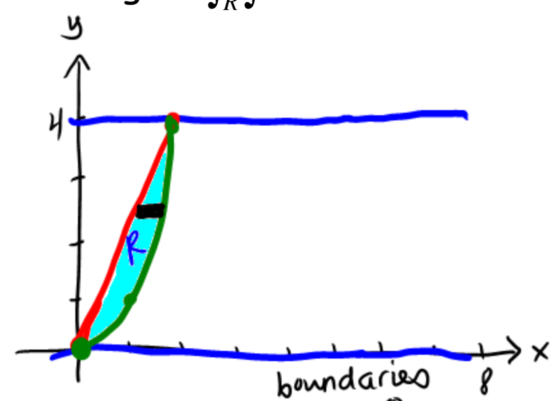
$$= \int_0^4 y^2\left(\frac{y^{3/2}}{3} - \frac{y^3}{24}\right)dy$$

$$= \int_0^4 \left(\frac{y^{7/2}}{3} - \frac{y^5}{24}\right)dy$$

$$= \left(\frac{2y^{9/2}}{27} - \frac{y^6}{144}\right)\Big|_0^4$$

$$= \boxed{\frac{2}{27}(512) - \frac{4096}{144}} - 0$$

$$= \boxed{\frac{256}{27}\text{ units}^3}$$



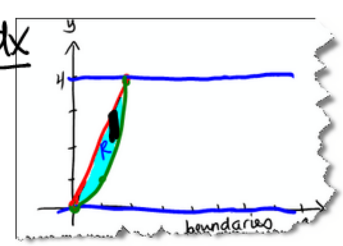
$0\le y\le 4 \to y=0$, $y=4$

$\frac{1}{2}y \le x \le \sqrt{y} \to x=\frac{1}{2}y \to y=2x$

$x=\sqrt{y} \to y=x^2$

Set up as dydx

$$\int_0^2\int_{x^2}^{2x} x^2y^2\,dy\,dx$$



PROPERTIES OF DOUBLE INTEGRALS

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\int_R\int cf(x,y)\,dA = c\int_R\int f(x,y)\,dA$
2. $\int_R\int\left[f(x,y)\pm g(x,y)\right]dA = \int_R\int f(x,y)\,dA \pm \int_R\int g(x,y)\,dA$
3. $\int_R\int f(x,y)\,dA \geq 0, \quad \text{if } f(x,y)\geq 0$
4. $\int_R\int f(x,y)\,dA \geq \int_R\int g(x,y)\,dA, \quad \text{if } f(x,y)\geq g(x,y)$
5. $\int_R\int f(x,y)\,dA = \int_{R_1}\int f(x,y)\,dA + \int_{R_2}\int f(x,y)\,dA$, where R is the union of two nonoverlapping subregions R_1 and R_2

THEOREM: FUBINI'S THEOREM

Let f be continuous on a plane region R.

1. If R is defined by $a \le x \le b$ and $g_1(x)\le y\le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then
$$\int_R\int f(x,y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} dydx$$
2. If R is defined by $c\le y\le d$ and $h_1(y)\le x\le h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then
$$\int_R\int f(x,y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} dxdy$$

Example 2: Set up an integrated integral for both orders of integration, and use the more convenient order to evaluate over the region R.

$$\int_R\int xe^y\,dA,$$

R: triangle bounded by $y=4-x,\ y=0,\ x=0$

$\int xe^{4-x}dx$

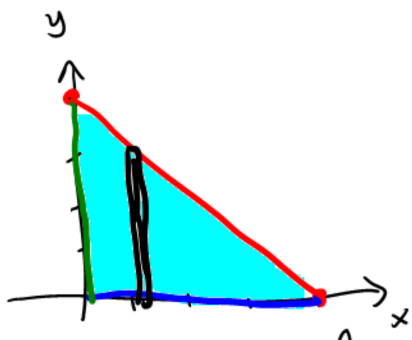
$u=x,\ du=dx$

$dv=e^{4-x}dx$

$v=-e^{4-x}$

$= -xe^{4-x}+\int e^{4-x}dx$

$= -xe^{4-x}-e^{4-x}$



vertically simple

$$\int_0^4\int_0^{4-x} xe^y\,dydx = \int_0^4 xe^y\Big|_0^{4-x}dx$$

$$= \int_0^4 x\left(e^{4-x}-e^0\right)dx$$

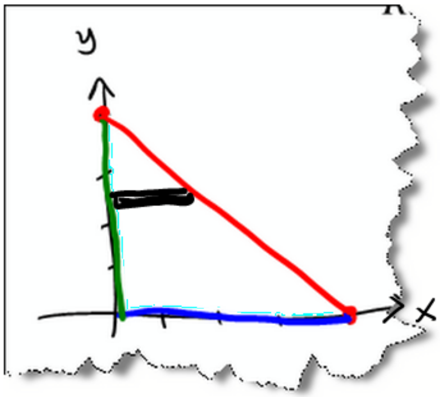
$$= \int_0^4\left(xe^{4-x}-x\right)dx$$

$$= -xe^{4-x}-e^{4-x}-\frac{x^2}{2}\Big|_0^4$$

$$= \left(-4e^0-e^0-\frac{16}{2}\right)-\left(-0e^4-e^4\right)$$

$$= -5-8+e^4$$

$$= e^4-13$$



horizontally simple

$$\int_0^4 \int_0^{4-y} x e^y \, dx\, dy$$

If $y=4-x$,
$x = 4-y$

Harder to integrate!

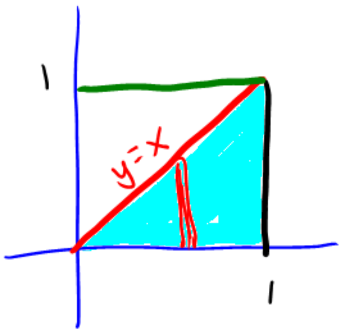
Example 3: Set up a double integral to find the volume of the solid bounded by the graphs of the equations $x^2 + z^2 = 1,\ y^2 + z^2 = 1,$ first octant.

$z = \sqrt{1-x^2},\ z = \sqrt{1-y^2}$ $\quad z \geq 0$

$\sqrt{1-x^2} = \sqrt{1-y^2}$

$x = y$

if $z = 0 \to x = y = 1$



$V = 2\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx$

$V = 2\int_0^1 \sqrt{1-x^2} \left(y \Big|_0^x \right) dx$

$V = \frac{2}{-2} \int_0^1 -2x\sqrt{1-x^2}\, dx$

$V = -\frac{2}{3}(1-x^2)^{3/2} \Big|_0^1$

$V = -\frac{2}{3}(0-1)$

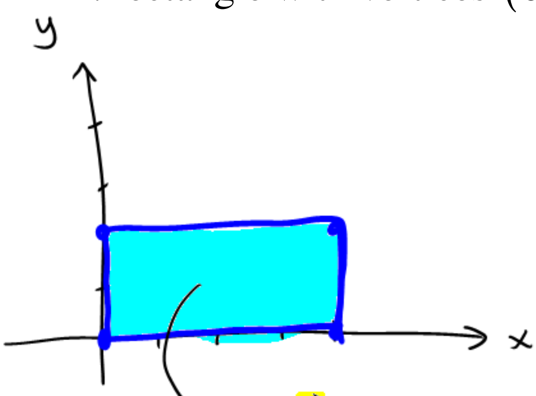
$\boxed{V = \frac{2}{3} \text{ cubic units}}$

Example 4: Find the average value of $f(x, y)$ over the region R where

$$\text{Average value} = \frac{1}{A} \int_R \int f(x, y)\, dA, \text{ where } A \text{ is the area of } R.$$

$f(x, y) = xy.$

R: rectangle with vertices $(0,0)$, $(4,0)$, $(4,2)$ and $(0,2)$.



$A = 4 \cdot 2 = 8$

$$\text{Average Value} = \frac{1}{8} \int_0^4 \int_0^2 xy\, dy\, dx$$

$$= \frac{1}{8} \int_0^4 x\frac{y^2}{2} \Big|_0^2 dx$$

$$= \frac{1}{16} \int_0^4 x(2^2 - 0^2)\, dx$$

$$= \frac{1}{4} \int_0^4 x\, dx$$

$$= \frac{1}{4} \cdot \frac{x^2}{2} \Big|_0^4$$

$$= \frac{1}{8}(4^2 - 0^2)$$

$$= \boxed{2}$$