4/4/11	Wednesday	Friday
· Finish 14.2 · Lecture 14.3	· Finish 14.3 · Lecture 14.5	Lecture 14.6

When you are done with your homework you should be able to…

π Write and evaluate double integrals in polar coordinates

Warm-up: Find the area of the region inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$..

① Intersection angles

$3\sin\theta = 2 - \sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

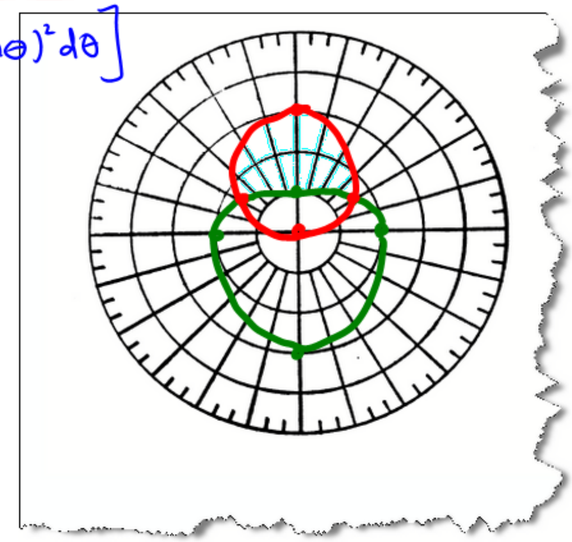
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2}(2-\sin\theta)^2\,d\theta\right]$

$A = \int_{\pi/6}^{\pi/2}\left(9\sin^2\theta - 4 + 4\sin\theta - \sin^2\theta\right)d\theta$

$A = \int_{\pi/6}^{\pi/2}\left(8\sin^2\theta + 4\sin\theta - 4\right)d\theta$

$A = 4\int_{\pi/6}^{\pi/2}\left(2\sin^2\theta + \sin\theta - 1\right)d\theta$



Recall: You guys can finish !! ☺

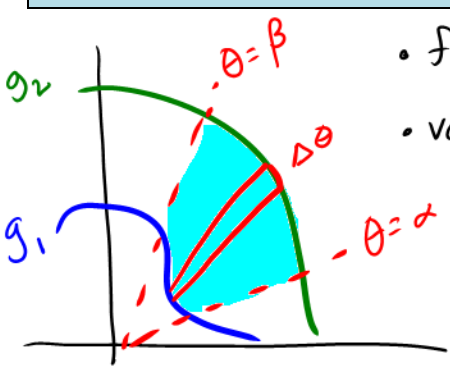
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \ \text{ and } \ \tan\theta = \frac{y}{x}$$

THEOREM: CHANGE OF VARIABLES IN POLAR FORM

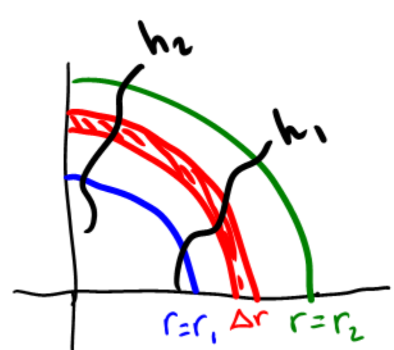
Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\int_R\int f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



r-simple region

- fixed bounds for θ
  $\alpha \le \theta \le \beta$
- variable bounds for r
  $0 \le g_1(\theta) \le r \le g_2(\theta)$



θ-simple region

Variable bounds for θ

$0 \le h_1(r) \le \theta \le h_2(r)$

Fixed bounds for r

$r_1 \le r \le r_2$

Example 1: Evaluate the double integral $\int_R\int f(r,\theta)\,dA$ and sketch the region R.

$$\int_0^{\pi/4}\int_0^4 r^2\sin\theta\cos\theta\,dr\,d\theta$$

$$=\int_0^{\pi/4}\sin\theta\cos\theta\left(\frac{r^3}{3}\Big|_0^4\right)d\theta$$

$$=\frac{64}{3}\left(\frac{\sin^2\theta}{2}\Big|_0^{\pi/4}\right.$$

$$=\frac{32}{3}\left[(\sin\pi/4)^2-(\sin 0)^2\right]$$

$$=\frac{32}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^2-(0)^2\right]$$

$$=\frac{32}{3}\left(\frac{1}{2}\right)$$

$$=\boxed{\frac{16}{3}}$$

Example 2: Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^2\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\,dx\,dy$$

$x=y$, $x=\sqrt{8-y^2}$

$x^2=8-y^2$

$x^2+y^2=8$

$r^2=8$

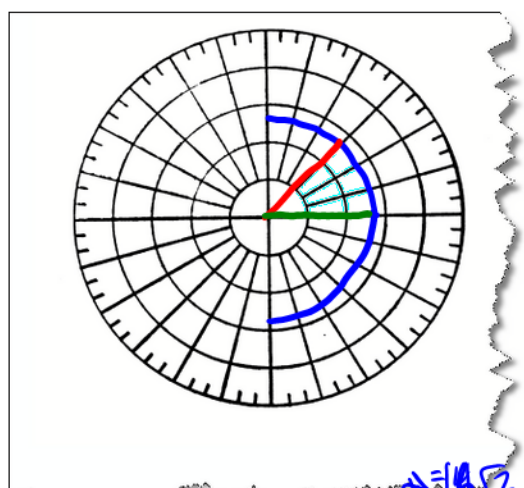
$r=2\sqrt{2}$

circle w/radius $2\sqrt{2}\approx 2.8$

and center at the pole

$y=0$, $y=2$

$$\begin{bmatrix} 0\le y\le 2 \\ y\le x\le\sqrt{8-y^2}\end{bmatrix}$$



$y=x$

$r\sin\theta=r\cos\theta$

$r(\sin\theta-\cos\theta)=0$

$r=0$ or $\sin\theta=\cos\theta$

$\theta=\frac{\pi}{4}$

$$\int_0^{\pi/4}\int_0^{2\sqrt{2}} r\cdot r\,dr\,d\theta=\int_0^{\pi/4}\frac{r^3}{3}\Big|_0^{2\sqrt{2}}d\theta$$

formula

$$=\frac{16\sqrt{2}}{3}\int_0^{\pi/4}d\theta$$

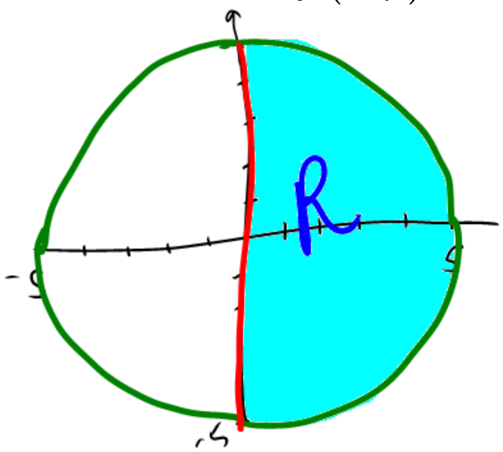
$$=\frac{16\sqrt{2}}{3}\theta\Big|_0^{\pi/4}$$

$$=\frac{16\sqrt{2}}{3}\cdot\frac{\pi}{4}$$

$$=\boxed{\frac{4\sqrt{2}}{3}\pi}$$

Example 3: Use polar coordinates to set up and evaluate the double integral $\int_R \int f(x,y)\,dA$.

$$f(x,y) = e^{-\left(x^2+y^2\right)/2},\ R: x^2+y^2 \le 25,\ x^2 \ge 0$$



$r^2 \le 25$ | $x \ge 0$

$r \le 5$

$$\iint_R f(x,y)\,dA = -\int_{-\pi/2}^{\pi/2}\int_0^5 -r \cdot e^{-r^2/2}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} e^{-r^2/2}\Big|_0^5\,d\theta$$

$$= -\int_{-\pi/2}^{\pi/2} \left(e^{-25/2} - 1\right)d\theta$$

$$= \left(e^{-25/2} - 1\right)\theta\Big|^{\pi/2}_{-\pi/2}$$

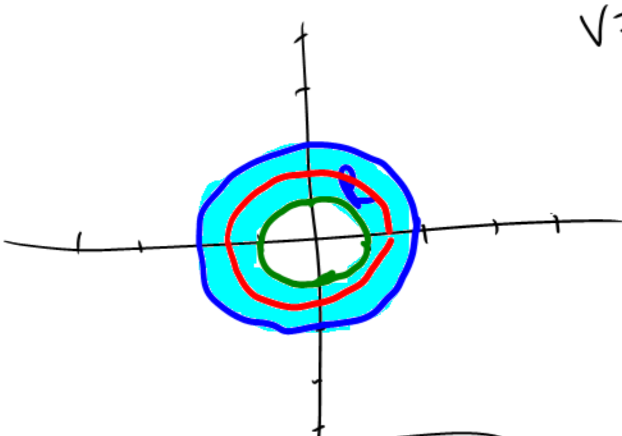
$$= \left(1 - e^{-25/2}\right)\left(\pi/2 - \left(-\tfrac{\pi}{2}\right)\right)$$

$$= \boxed{\pi\left(1 - e^{-25/2}\right)}$$

Example 4: Use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations

$$z = \ln\left(x^2+y^2\right),\ z = 0,\ x^2+y^2 \ge 1,\ x^2+y^2 \le 4$$

$$V = \iint_R f(x,y)\,dA = \int_0^{2\pi}\int_1^2 r \cdot \ln r^2\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_1^2 2r\ln r\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(8\ln 2 - 3)\,d\theta$$

$$= \tfrac{1}{2}(8\ln 2 - 3)\theta\Big|_0^{2\pi}$$

$$= \tfrac{1}{2}(8\ln 2 - 3)(2\pi - 0)$$

$$= \boxed{\pi(8\ln 2 - 3)}$$

$u = \ln r$, $dv = 2r\,dr$

$du = \frac{dr}{r}$, $v = r^2$

$$\int_1^2 2r\ln r\,dr = r^2\ln r - \int r^{\cancel{2}} \cdot \frac{dr}{\cancel{r}}$$

$$= \left(r^2\ln r - \frac{r^2}{2}\right)\Big|_1^2$$

$$= \left(4\ln 2 - \tfrac{4}{2}\right) - \left(1\ln 1 - \tfrac{1}{2}\right)$$

$$= 4\ln 2 - \tfrac{3}{2}$$

$$= \tfrac{1}{2}\left(8\ln 2 - 3\right)$$