EXAM 5/CHAPTER 15.1-15.5
75 POINTS POSSIBLE

NAME Key

LEAVE ALL ANSWERS EXACT UNLESS THE PROBLEM INDICATES OTHERWISE
SHOW ALL WORK IN ORDER TO EARN FULL CREDIT

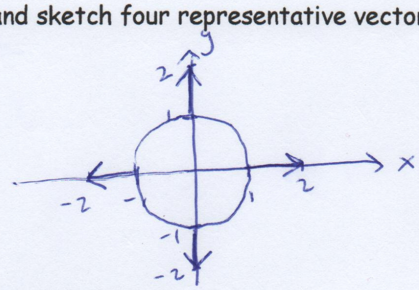
1. (10 POINTS) Find $\|\mathbf{F}\|$ and sketch four representative vectors in the vector field $\mathbf{F}(x,y)=x\mathbf{i}+y\mathbf{j}$.

$\|\vec{F}\|=c$

$\|\vec{F}\|=\sqrt{x^2+y^2}$

so $x^2+y^2=c^2$

let $c=1$



Point	Vector
$(1,0)$	$\hat{i}$
$(0,1)$	$\hat{j}$
$(-1,0)$	$-\hat{i}$
$(0,-1)$	$-\hat{j}$

2. (15 POINTS) Consider the vector field $\mathbf{F}(x,y,z)=y^2z^3\mathbf{i}+2xyz^3\mathbf{j}+3xy^2z^2\mathbf{k}$.

 a. (6 POINTS) Show that $\mathbf{F}$ is conservative.

$M=y^2z^3$ | $N=2xyz^3$ | $P=3xy^2z^2$

$\frac{\partial M}{\partial y}=2yz^3$ | $\frac{\partial N}{\partial x}=2yz^3$ | $\frac{\partial P}{\partial x}=3y^2z^2$

$\frac{\partial M}{\partial z}=3y^2z^2$ | $\frac{\partial N}{\partial z}=6xyz^2$ | $\frac{\partial P}{\partial y}=6xyz^2$

$\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z}$ ✓

$\frac{\partial P}{\partial x}=\frac{\partial M}{\partial z}$ ✓

$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}$ ✓

$\boxed{\text{so } \vec{F} \text{ is conservative}}$

 b. (6 POINTS) Find a potential function for $\mathbf{F}$.

$\int M\,dx=\int y^2z^3\,dx=xy^2z^3+g(y,z)$

$\int N\,dy=\int 2xyz^3\,dy=xy^2z^3+h(x,z)$

$\int P\,dz=\int 3xy^2z^2\,dz=xy^2z^3+K(x,y)$

$g(y,z)=h(x,z)=K(x,y)=K$

$\boxed{f(x,y,z)=xy^2z^3+K}$

 c. (3 POINTS) Find $\operatorname{div}\mathbf{F}$.

$\operatorname{div}\vec{F}=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$

$=0+2xz^3+6xy^2z$

$=\boxed{2xz(z^2+3y^2)}$

3. (10 POINTS) Evaluate the line integral along the given path.

$$\int_C 2xyz\,ds$$

$C: \mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 84t\mathbf{k},\ 0 \le t \le 1$

$x(t)=12t, x'(t)=12$
$y(t)=5t, y'(t)=5$
$z(t)=84t, z'(t)=84$

$f(x(t),y(t),z(t))$
$= f(12t, 5t, 84t)$
$= 2(12t)(5t)(84t)$
$= 10080t^3$

$\int_C 2xyz\,ds = \int_0^1 10080t^3\sqrt{(12)^2+(5)^2+(84)^2}\,dt$

$= \int_0^1 10080t^3\sqrt{7225}\,dt$

$= 856800\int_0^1 t^3\,dt$

$= 856800\left(\frac{t^4}{4}\Big|_0^1\right.$

$= 214200(1^4 - 0^4)$

$= \boxed{214{,}200}$

4. (10 POINTS) Use the Fundamental Theorem of Line Integrals to evaluate the line integral $\int_C \frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, where C is the line segment from $(1,1)$ to $(2\sqrt{3},2)$.

$M = \frac{y}{x^2+y^2}$, $N = +\frac{x}{x^2+y^2}$

$\int M\,dx = \int \frac{y}{x^2+y^2}dx = y\left(\frac{1}{y}\arctan\frac{x}{y}\right)$

$= \arctan\frac{x}{y}$

$\int N\,dy = \int \frac{x}{x^2+y^2}dy = x\left(\frac{1}{x}\arctan\frac{y}{x}\right)$

$= \arctan\frac{x}{y}$

$\left(\arctan\frac{x}{y} + \arctan\frac{y}{x}\right|_{(1,1)}^{(2\sqrt{3},2)}$

$= \left(\arctan\frac{2\sqrt{3}}{2} + \arctan\frac{2}{2\sqrt{3}}\right) - \left(\arctan\frac{1}{1} + \arctan\frac{1}{1}\right)$

$= \frac{\pi}{3} + \frac{\pi}{6} - \frac{\pi}{4} - \frac{\pi}{4}$

$= \boxed{0}$

5. (10 POINTS) Use Green's Theorem to evaluate the line integral.

$$\int_C (y-x)\,dx + (2x-y)\,dy$$

$C: x = 3\cos\theta,\ y = 5\sin\theta$

$M = y - x \qquad N = 2x - y$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 2$

$\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ ellipse $a=5, b=3$

Area $= \pi ab$
$= \pi \cdot 15$

$\int_C (y-x)\,dx + (2x-y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$

$= \iint_R 1\,dA$

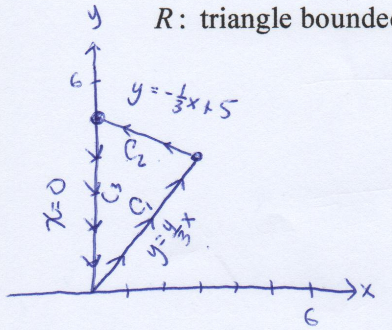
$= (1)(15\pi)$

$= \boxed{15\pi}$

6. (10 POINTS) Use a line integral to find the area of the region R. $y = -\frac{1}{3}x + 5$

R: triangle bounded by the graphs of $x = 0,\ 4x - 3y = 0,$ and $x + 3y = 15$

$y = \frac{4}{3}x$



$C_1: y = \frac{4}{3}x,\ dy = \frac{4}{3}dx,\ 0 \le x \le 3$

$C_2: y = -\frac{1}{3}x + 5,\ dy = -\frac{1}{3}dx,\ 3 \le x \le 0$

$C_3: x = 0,\ dx = 0,\ 0 \le x \le 0$

$$A = \frac{1}{2}\int x\,dy - y\,dx$$

$$A = \frac{1}{2}\int_0^3 \left[x\left(\frac{4}{3}dx\right) - \left(\frac{4}{3}x\right)dx\right] + \frac{1}{2}\int_3^0 \left[\underbrace{x\left(-\frac{1}{3}dx\right) - \left(-\frac{1}{3}x + 5\right)dx}_{-\frac{1}{3}x + \frac{1}{3}x - 5}\right] + \frac{1}{2}\int_6^0 \left[0\,dy - y(0)\right]$$

$$A = 0 + \frac{1}{2}\int_3^0 (-5)\,dx + 0$$

$$A = -\frac{5}{2}x\Big|_3^0 = -\frac{5}{2}(0 - 3) = \boxed{\frac{15}{2}\text{ sq. units}}$$

7. (10 POINTS) Find an equation of the tangent plane to the surface represented by the vector-valued function $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\mathbf{k}$ at the point $(1,1,1)$.

$$\vec{r}_u = \hat{i} + \frac{u}{2\sqrt{uv}}\hat{k}$$

$$\vec{r}_v = \hat{j} + \frac{v}{2\sqrt{uv}}\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{u}{2\sqrt{uv}} \\ 0 & 1 & \frac{v}{2\sqrt{uv}} \end{vmatrix}$$

$$= \left(0 - \frac{u}{2\sqrt{uv}}\right)\hat{i} - \left(\frac{v}{2\sqrt{uv}} - 0\right)\hat{j} + (1 - 0)\hat{k}$$

$$= -\frac{u}{2\sqrt{uv}}\hat{i} - \frac{v}{2\sqrt{uv}}\hat{j} + \hat{k}$$

at $(1,1,1)$:

$$\vec{r}_u \times \vec{r}_v = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + k$$

$$= -\frac{1}{2}\langle 1, 1, -2\rangle$$

$$1(x-1) + 1(y-1) - 2(z-1) = 0$$

$$x - 1 + y - 1 - 2z + 2 = 0$$

$$\boxed{x + y - 2z = 0}$$