When you are done with your homework you should be able to…

- π Find the cross product of two vectors in space
- π Use the triple scalar product of three vectors in space

Warm-up: Find the direction cosines of $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and demonstrate that the sum of the squares of the direction cosines is 1.

DEFINITION OF CROSS PRODUCT OF TWO VECTORS IN SPACE

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space.

The cross product of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

THEOREM: ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = (c\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(c\mathbf{v})$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
5. $\mathbf{u}\times\mathbf{u} = \mathbf{0}$
6. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

THEOREM: GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$.
3. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u}\times\mathbf{v}\| =$ the area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Example 1: Find $\mathbf{u}\times\mathbf{v}$ and show that it is orthogonal to both $\mathbf{u} = \langle -1,1,2\rangle$ and $\mathbf{v} = \langle 0,1,0\rangle$.

In physics, the cross product can be used to measure **torque**, which is the moment $\mathbf{M}$ of a force $\mathbf{F}$ about a point *P*. If the point of application of the force is *Q*, the moment of $\mathbf{F}$ about *P* is given by $\mathbf{M}=\overrightarrow{PQ}\times\mathbf{F}$. The magnitude of the moment $\mathbf{M}$ measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise about an axis directed along the vector $\mathbf{M}$.

Example 2: Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

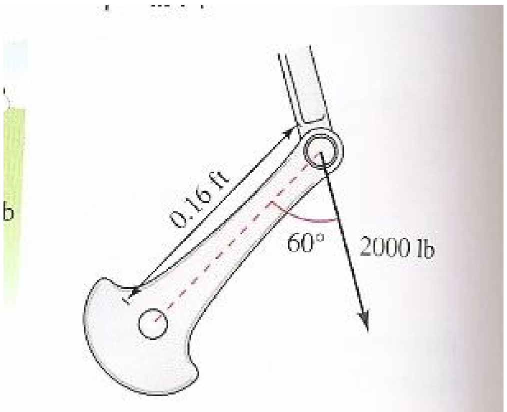

Figure for 38

THEOREM: THE TRIPLE SCALAR PRODUCT

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$,
The triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 3: Find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. $\mathbf{u} = \langle 1,1,1 \rangle$, $\mathbf{v} = \langle 2,1,0 \rangle$, $\mathbf{w} = \langle 0,0,1 \rangle$.

Example 4: Find the volume of the parallelepiped having adjacent edges $\mathbf{u} = \langle 1,3,1 \rangle$, $\mathbf{v} = \langle 0,6,6 \rangle$, $\mathbf{w} = \langle -4,0,-4 \rangle$.